Mots sturmiens images par morphismes de points fixes de morphismes

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3 juillet 2017

Journées SDA2 2017 - Marseille

Sturmian words

- Infinite words over $\mathcal{A} = \{a, b\}$;
- Non ultimately periodic ($\neq uv^\omega$, $u \in \mathcal{A}^*$, $v \in \mathcal{A}^+$) ;
- $|u| = |v| \Rightarrow ||u|_a - |v|_a| \leq 1$ ($\Rightarrow ||u|_b - |v|_b| \leq 1$).
  $\Rightarrow$ a Sturmian word cannot contain both a^2 and b^2.

Lothaire, Proposition 2.1.3

If an infinite word x over $\mathcal{A}$ is not Sturmian then it contains the two factors awa and bwb for some w.

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Sturmian morphisms

- A morphism f on $\mathcal{A}$ is *Sturmian* if $f(x)$ is Sturmian whenever x is a Sturmian word.

- $St = \{E, \varphi, \tilde{\varphi}\}^*$

$$E: \begin{array}{ccc} a & \mapsto & b \\ b & \mapsto & a \end{array} \qquad \varphi: \begin{array}{ccc} a & \mapsto & ab \\ b & \mapsto & a \end{array} \qquad \tilde{\varphi}: \begin{array}{ccc} a & \mapsto & ba \\ b & \mapsto & a \end{array}$$

  f is a Sturmian morphism if and only if $f \in St$.

- A Sturmian morphism f generates a Sturmian word iff $f \in St \setminus (\{Id, E\} \cup \{\varphi E, \tilde{\varphi} E\}^+ \cup \{E\varphi, E\tilde{\varphi}\}^+)$.

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If x is a Sturmian word then $E(x)$, $\varphi(x)$, $\tilde{\varphi}(x)$ are Sturmian words.

Lothaire, Proposition 2.3.2

Let x be an infinite word over $\mathcal{A}$.

(i) If $E(x)$ is Sturmian then x is Sturmian.

(ii) If $\varphi(x)$ is Sturmian then x is Sturmian.

(iii) If $\tilde{\varphi}(x)$ is Sturmian

- if x starts with the letter a then x is Sturmian.
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Example : The Fibonacci word F

$F = \varphi^\omega(a) = \lim_{n\to\infty} \varphi^n(a) = abaababaabaababaaba\cdots$

F is Sturmian ; aF and bF are Sturmian.

Property. For every infinite word x over $\mathcal{A}$, $a\tilde{\varphi}(x) = \varphi(x)$.

$\to \tilde{\varphi}(bF) = a\tilde{\varphi}(F) = \varphi(F) = F$.
$\to \tilde{\varphi}(bbF) = a\tilde{\varphi}(bF) = aF$ is Sturmian.

bbF is not Sturmian because F contains aa
but $bbF = bx'$ where $x' = bF$ is Sturmian.
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Lothaire, Theorem 2.3.7

Let x be a Sturmian word and f a morphism on $\mathcal{A}$. If $f(x)$ is Sturmian then $f \in St$.

Proposition

Let x be an infinite word over $\mathcal{A}$ and f a morphism on $\mathcal{A}$. If $f(x)$ is Sturmian then $f \in St$.

Theorem 1

Let x be an infinite word over $\mathcal{A}$ and g a morphism on $\mathcal{A}$ such that $x = g(x)$. If x is Sturmian then $g \in St$ and if, moreover, $g \neq Id_{\mathcal{A}}$ then there exists $c \in \mathcal{A}$ such that $x = g^{\omega}(c)$.

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Sturmian fixed points of morphisms

Corollary

If a Sturmian word x over $\mathcal{A}$ is a fixed point of some morphism $g \neq Id_{\mathcal{A}}$ then

- x is generated by the D0L system $G = (\mathcal{A}, g, c)$ where $g \in St$ is prolongable on the letter $c \in \mathcal{A}$ ($x = g^{\omega}(c)$) ;
- $g \in St \setminus (\{Id, E\} \cup \{\varphi E, \tilde{\varphi} E\}^{+} \cup \{E\varphi, E\tilde{\varphi}\}^{+})$;
- there exists a morphism $h \in St$ such that x is rigid of base h (each morphism of which x is a fixed point is a power of h).

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Proposition

Let x be an infinite word over $\mathcal{A}$. Are equivalent :

$\exists\ f$ morphism on $\mathcal{A}$ such that $f(x)$ is Sturmian

$\Leftrightarrow \quad \exists\ n \in \mathbb{N},\ c \in \mathcal{A},\ x'$ Sturmian word such that $x = c^n x'$.

B. Parvaix,
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PhD thesis, (1998).

$\forall\ n \in \mathbb{N},\ n \geq 1,\ \exists\ x$ Sturmian word such that a^2 is a factor of x : thus b^2x is not Sturmian but $(\tilde{\varphi}E)^n\tilde{\varphi}(b^{n+1}x) = a^{n+1}(\tilde{\varphi}E)^n\tilde{\varphi}(x) = (\varphi E)^n\varphi(x)$ is Sturmian for every $n \in \mathbb{N}$.

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Theorem 2

Let y be a Sturmian word over $\mathcal{A}$, and f, g morphisms on $\mathcal{A}$, x infinite word over $\mathcal{A}$ such that $y = f(x)$ and $x = g(x)$:

- $f \in St$ and $g \in St$;
- *exists* $n \in \mathbb{N}$ and a Sturmian word x' such that $x = a^n x'$ or $x = b^n x'$;
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St is left unitary : $f \in St$ and $fg \in St \Rightarrow g \in St$.

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Corollary

If a Sturmian word y over $\mathcal{A}$ is the image by a morphism f on $\mathcal{A}$ of a fixed point of a morphism $g \neq Id_{\mathcal{A}}$ then y is generated by the HD0L system $H = (\mathcal{A}, g, f, c)$ where the morphisms f and g are Sturmian and $y = f(g^{\omega}(c))$ where $g^{\omega}(c)$ is the Sturmian word generated by the D0L system $(\mathcal{A}, g, c)$.

Proposition

The word $\tilde{\varphi}(F)$ fulfills the conditions of Theorem 2 but not those of Theorem 1 (with $g \neq Id_{\mathcal{A}}$).

$\tilde{\varphi}(F) = \tilde{\varphi}(\varphi^{\omega}(a))$ is generated by the HD0L system $(\mathcal{A}, \varphi, \tilde{\varphi}, a)$.
$\tilde{\varphi}(F) = a^{-1}F$ is not generated by a morphism.

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For $c \in \mathcal{A}$, let $\bar{c}$ be such that $\{c, \bar{c}\} = \mathcal{A}$.

Let NS = { x infinite word over $\mathcal{A}$ / x = $\mathrm{c}^k \bar{c} x'$
with c $\in \mathcal{A}, k \geq 1, \bar{c}x'$ non Sturmian }.

Lemma

Let $x \in NS$ and $f \in St$. Then $f(x) \in NS$.

Let $S = \{$Sturmian words over $\mathcal{A}\}$.

Let $S_{c^n} = \{c^n x, x$ Sturmian and cx non Sturmian$\}$, $c \in \mathcal{A}$, $n \in \mathbb{N}$.

Let $X = \{$infinite words over $\mathcal{A}\}$.

$$X = S \cup NS \cup (\cup_{n \geq 1, c \in \mathcal{A}} S_{c^n})$$

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$$X = S \cup NS \cup (\cup_{n \geq 1, c \in \mathcal{A}} S_{c^n})$$

$$S \cap NS = S \cap (\cup_{n \geq 1, c \in \mathcal{A}} S_{c^n}) = NS \cap (\cup_{n \geq 1, c \in \mathcal{A}} S_{c^n}) = \emptyset.$$

For $c \in \mathcal{A}$, let $\bar{c}$ be such that $\{c, \bar{c}\} = \mathcal{A}$.

Let NS = { x infinite word over $\mathcal{A}$ / x = $c^k \bar{c} x'$
with c $\in \mathcal{A}, k \geq 1, \bar{c}x'$ non Sturmian }.

Lemma

Let $x \in NS$ and $f \in St$. Then $f(x) \in NS$.

Let $S = \{$Sturmian words over $\mathcal{A}\}$.

Let $S_{c^n} = \{c^n x, x$ Sturmian and cx non Sturmian$\}$, $c \in \mathcal{A}$, $n \in \mathbb{N}$.

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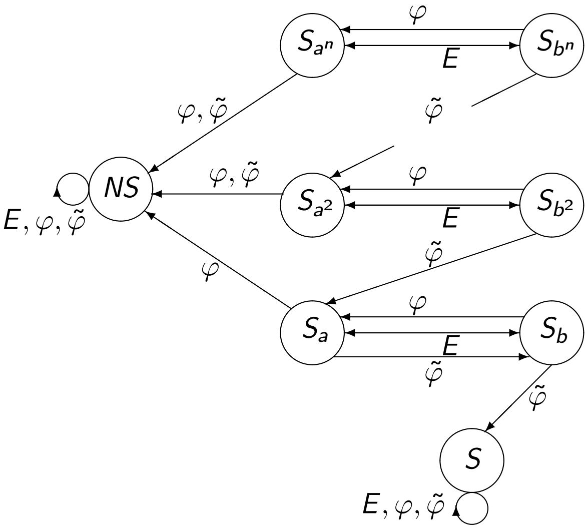
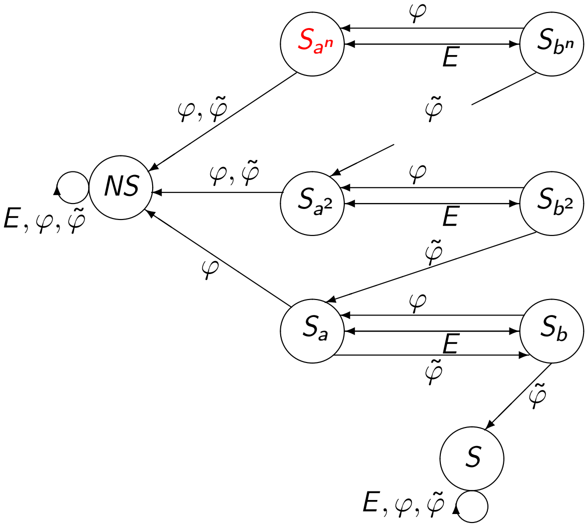
$$\mathrm{NS} = \{ \ \mathrm{x} \text{ infinite word over } \mathcal{A} \ / \ \mathrm{x} = \mathrm{c}^k \bar{c} x' \text{ with } \mathrm{c} \in \mathcal{A}, k \geq 1, \bar{c}x' \text{ non Sturmian } \}.$$

$S_{c^n} = \{c^n x, x$ Sturmian and cx non Sturmian$\}$, $c \in \mathcal{A}$, $n \in \mathbb{N}$.

Lemma

1. For every $f \in St$, $f(S) \subset S$ and $f(NS) \subset NS$.
2. For every $n \in \mathbb{N}$ and $c \in \mathcal{A}$, $E(S_{c^n}) = S_{\bar{c}^n}$.
3. For every $n \in \mathbb{N} \setminus \{0\}$, $\varphi(S_{b^n}) \subset S_{a^n}$ and $\tilde{\varphi}(S_{b^n}) \subset S_{a^{n-1}}$.
4. For every $n \in \mathbb{N} \setminus \{0\}$, $\varphi(S_{a^n}) \subset NS$.
5. For every $n \in \mathbb{N} \setminus \{0, 1\}$, $\tilde{\varphi}(S_{a^n}) \subset NS$.
6. $\tilde{\varphi}(S_a) \subset S_b$.


S_{a^n}
S_{b^n}
φ
E
$\varphi, \tilde\varphi$
$\tilde\varphi$
NS
$E, \varphi, \tilde\varphi$
$\varphi, \tilde\varphi$
S_{a^2}
S_{b^2}
φ
E
$\tilde\varphi$
φ
S_a
S_b
φ
E
$\tilde\varphi$
$\tilde\varphi$
S
$E, \varphi, \tilde\varphi$


S_{a^n}
S_{b^n}
φ
E
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$\varphi, \tilde{\varphi}$
S_{a^2}
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φ
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$E, \varphi, \tilde{\varphi}$
$\tilde{\varphi}$
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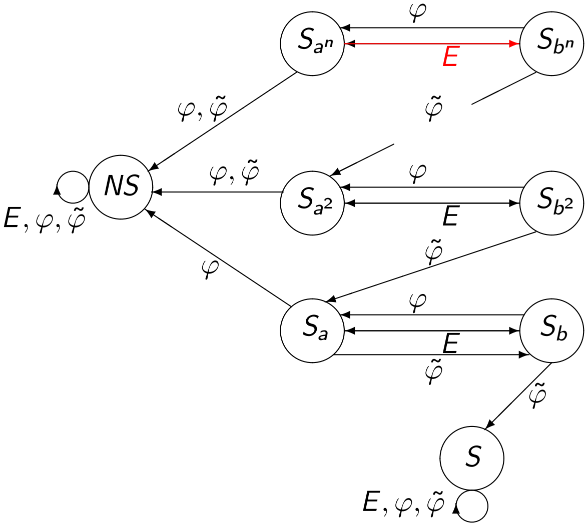
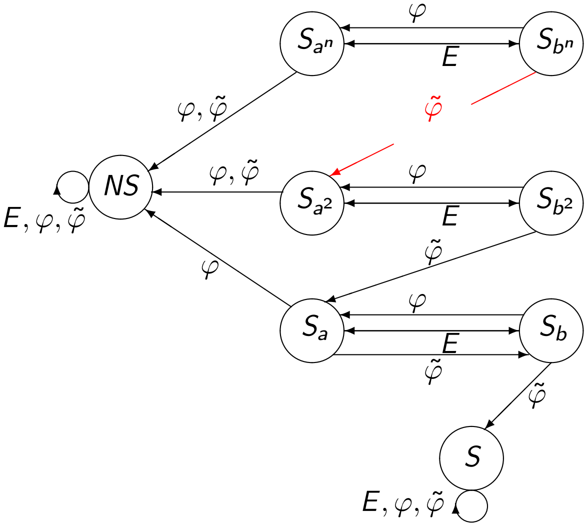
$\tilde{\varphi}$
$\tilde{\varphi}$
S
$E, \varphi, \tilde{\varphi}$

a^3x

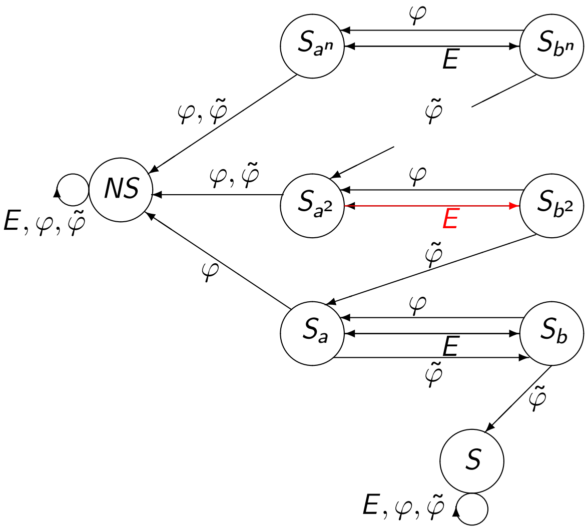


φ
S_{a^n}
S_{b^n}
E
$\varphi, \tilde{\varphi}$
$\tilde{\varphi}$
NS
$\varphi, \tilde{\varphi}$
φ
S_{a^2}
S_{b^2}
E
$E, \varphi, \tilde{\varphi}$
$\tilde{\varphi}$
φ
φ
S_a
S_b
E
$\tilde{\varphi}$
$\tilde{\varphi}$
S
$E, \varphi, \tilde{\varphi}$

$$E(a^3x)$$



$$\tilde\varphi E(a^3x)$$



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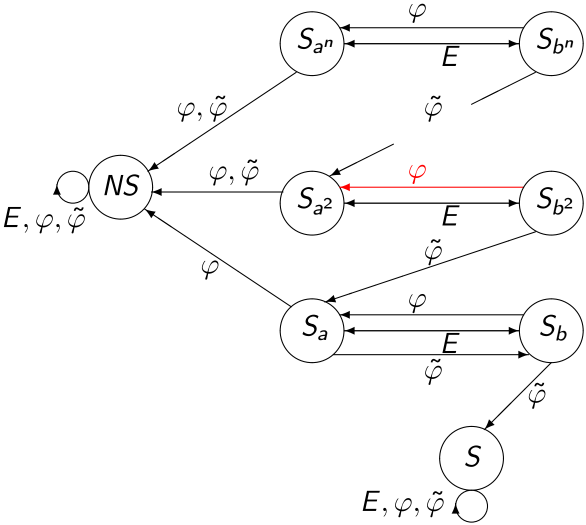
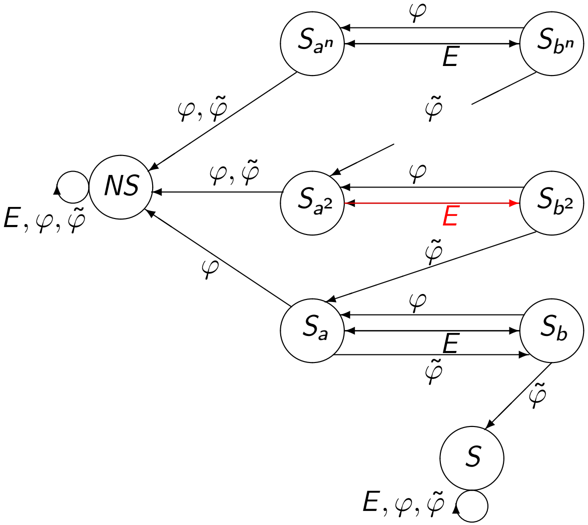
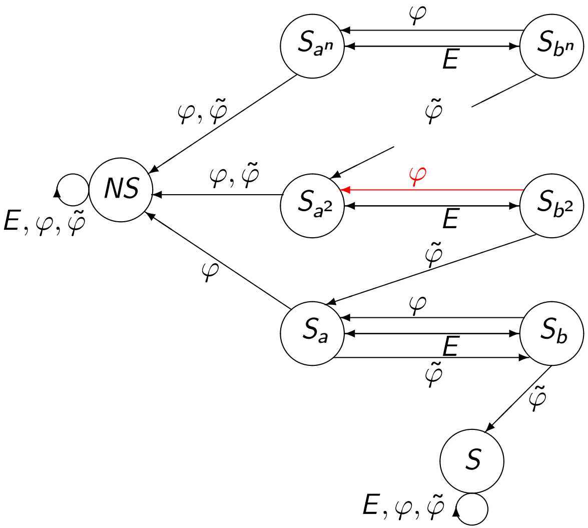


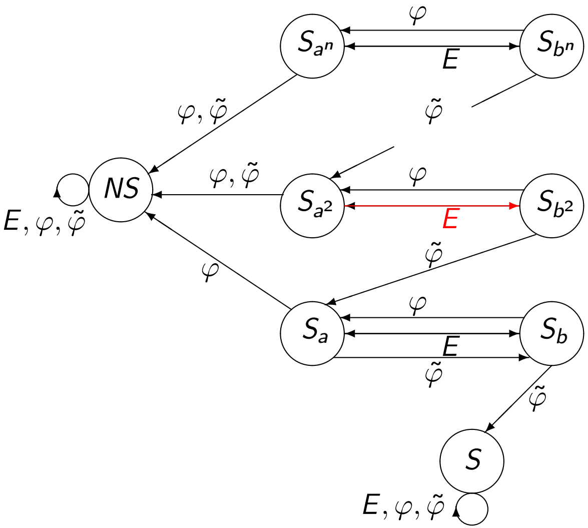
$$\varphi E \tilde\varphi E(a^3 x)$$



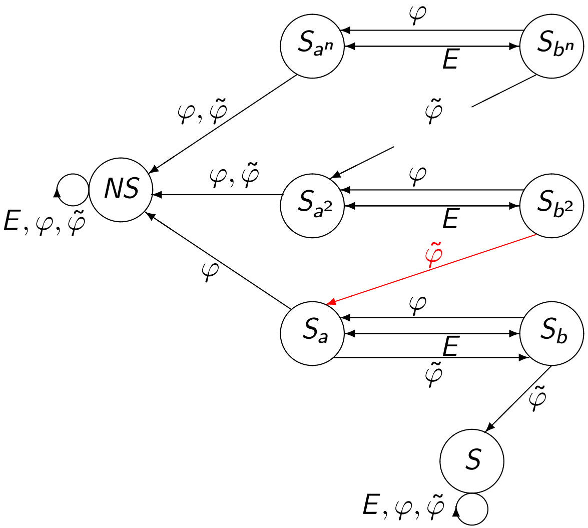
$$E\varphi E\tilde{\varphi}E(a^3x)$$



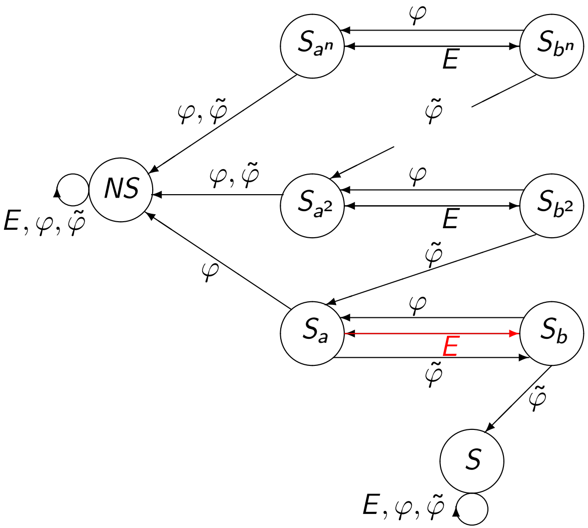
$$\varphi E \varphi E \tilde{\varphi} E(a^3x)$$



$$E\varphi E\varphi E\tilde{\varphi}E(a^3x)$$



$$\tilde{\varphi}(E\varphi)^2E\tilde{\varphi}E(a^3x)$$



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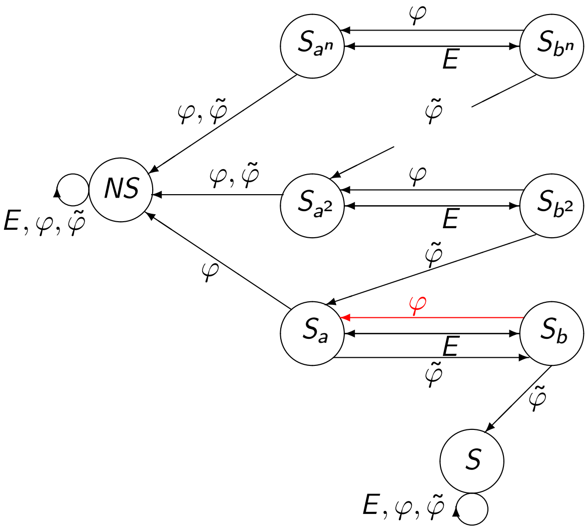
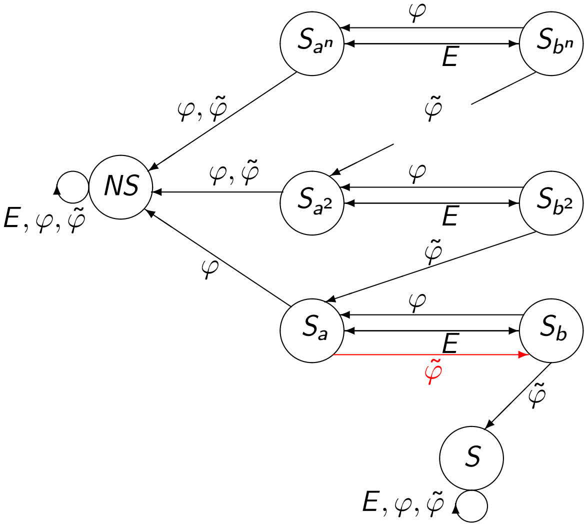
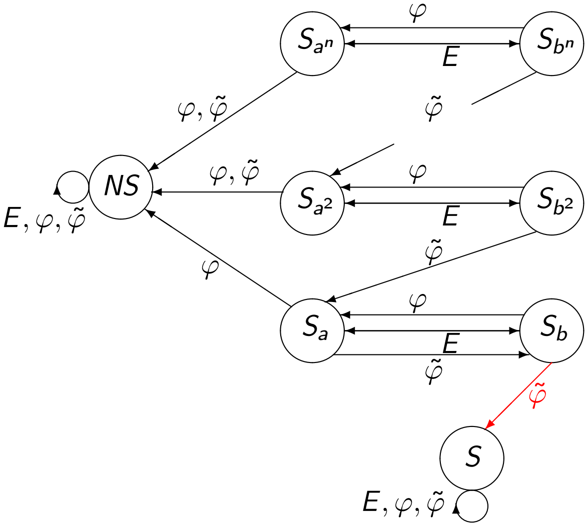


$$\varphi E\tilde{\varphi}(E\varphi)^2E\tilde{\varphi}E(a^3x)$$



$$\tilde{\varphi}\varphi E\tilde{\varphi}(E\varphi)^2E\tilde{\varphi}E(a^3x)$$



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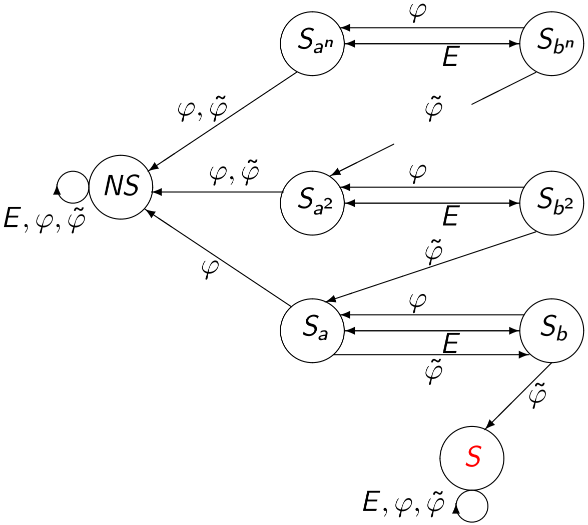
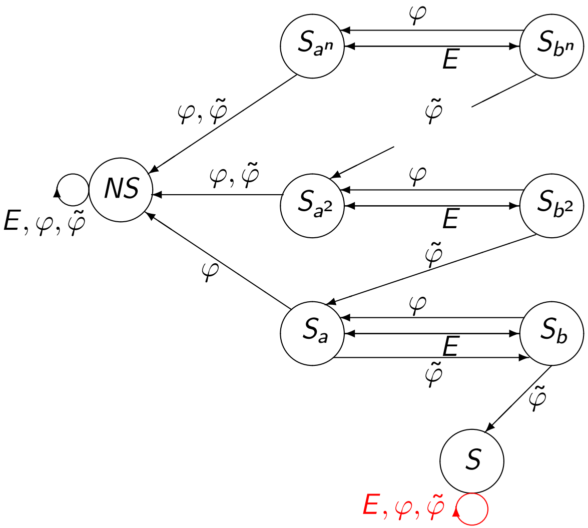


$$\tilde{\varphi}\tilde{\varphi}\varphi E\tilde{\varphi}(E\varphi)^2E\tilde{\varphi}E(a^3x) = \varphi^2(\varphi E)^5(x)$$



$$(E,\varphi,\tilde\varphi)^*\tilde\varphi\tilde\varphi\varphi E\tilde\varphi(E\varphi)^2E\tilde\varphi E(a^3x) = (E,\varphi,\tilde\varphi)^*\varphi^2(\varphi E)^5(x)$$

Theorem 3

Let x be a Sturmian word and let f be a morphism on $\mathcal{A}$.

- $f(x)$ is Sturmian if and only if $f \in St$;
- if bx is not Sturmian, $f(bx)$ is Sturmian if and only if $f = f'\tilde{\varphi}$, $f' \in St$;
- if ax is not Sturmian, $f(ax)$ is Sturmian if and only if $f = f'g$, $f' \in St$ and $g \in \tilde{\varphi}(E, \tilde{\varphi})$;
- if bx is not Sturmian, $f(b^n x)$ is Sturmian for some integer $n \geq 2$ if and only if $f = f'g$, $f' \in St$ and $g \in \tilde{\varphi}(E, \tilde{\varphi})(E\tilde{\varphi}, \varphi\tilde{\varphi}, \varphi E)^* \tilde{\varphi} \left[(E\varphi)^* E\tilde{\varphi}\right]^{n-2} (E\varphi)^*$;
- if ax is not Sturmian, $f(a^n x)$ is Sturmian for some integer $n \geq 2$ if and only if $f = f'g$, $f' \in St$ and $g \in \tilde{\varphi}(E, \tilde{\varphi})(E\tilde{\varphi}, \varphi\tilde{\varphi}, \varphi E)^* \tilde{\varphi} \left[(E\varphi)^* E\tilde{\varphi}\right]^{n-2} (E\varphi)^* E$.

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A morphism $f \in St$ is *standard* if $|f|_{\tilde{\varphi}} = 0$.

The writing of a morphism over $\{E, \varphi, \tilde{\varphi}\}$ is not unique.

Example : $\varphi^2\tilde{\varphi} = \tilde{\varphi}^2\varphi$

But $E^2 = Id$ and $\varphi(\varphi E)^k E\tilde{\varphi} = \tilde{\varphi}(\tilde{\varphi}E)^k E\varphi$, $k \in \mathbb{N}$,
is a presentation of St.

$\Rightarrow$ If a writing of $f \in St$ contains at least one $\tilde{\varphi}$, then all the writings of f contain at least one $\tilde{\varphi}$.

The writing (without E^2) of a standard morphism is unique.

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f morphism on $\mathcal{A}$ such that $f(a)$ and $f(b)$ begin with the same letter $c \in \mathcal{A}$.

The *conjugate* of f is the morphism g defined by

$$g(x) = c^{-1}f(x)c \text{ for all } x \in \mathcal{A}.$$

If f is standard then it has $|f(ab)| - 1$ conjugates (including itself).

Lemma

Let f be a Sturmian morphism. There exists a unique standard morphism g of which f is a conjugate and

- $f = f_1 f_2 \cdots f_n$, $f_i \in \{E, \varphi, \tilde{\varphi}\}$, $1 \leq i \leq n$;
- $g = g_1 g_2 \cdots g_n$, $f_i \in \{E, \varphi\}$, $1 \leq i \leq n$;
- if $f_i = E$ or $f_i = \varphi$ then $g_i = f_i$; if $f_i = \tilde{\varphi}$ then $g_i = \varphi$.

P. Séébold,
On the conjugaison of standard morphisms
Theoret. Comput. Sci., 195 (1998), 91-109.

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Let $R : St \to St$, the transformation defined by $R(E) = E$, $R(\tilde{\varphi}) = R(\varphi) = \varphi$.

For any $f \in St$, $R(f)$ is called the *standard representation* of f : $R(f)$ is the standard morphism of which f is a conjugate.

Theorem 4

Let x be an infinite word over $\mathcal{A}$ and $n \in \mathbb{N}$, $n \geq 2$.

Let $g_1 \in \tilde{\varphi}(E, \tilde{\varphi})(E\tilde{\varphi}, \varphi\tilde{\varphi}, \varphi E)^* \tilde{\varphi} \left[(E\varphi)^* E\tilde{\varphi}\right]^{n-2} (E\varphi)^*$

and $g_2 \in \tilde{\varphi}(E, \tilde{\varphi})(E\tilde{\varphi}, \varphi\tilde{\varphi}, \varphi E)^* \tilde{\varphi} \left[(E\varphi)^* E\tilde{\varphi}\right]^{n-2} (E\varphi)^* E$.

Then $g_1(b^n x) = [R(g_1)](x)$ and $g_2(a^n x) = [R(g_2)](x)$.

Merci pour votre attention !