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# Lattices generated by Chip Firing Game models: characterizations and recognition algorithm

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## Abstract

It is well-known that the class of lattices generated by CFGs is strictly included in the class of upper locally distributive lattices. However a full characterization for this class is still an open question. In this paper, we settle this problem by giving a full condition for this class. This condition allows us to build a polynomial-time algorithm for constructing a CFG which generates a given lattice if such a CFG exists. Going further, we solve the same problem on two other classes of lattices which are generated by CFGs on the classes of undirected graphs and directed acyclic graphs.

*Keywords:* Abelian Sandpile model, Chip Firing Game, discrete dynamic model, lattice, Sandpile model, ULD lattice, linear programming

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## 1. Introduction

Chip Firing Game is a discrete dynamical model which was first defined by A. Bjorner, L. Lovasz and W. Shor while studying the ‘balancing game’ [5, 6, 18]. The model has various applications in many fields of science such as physics [9, 1], computer science [5, 6, 11], social science [2, 3] and mathematics [3, 16, 17].

The model is a game which consists of a directed multi-graph  $G$  (also called *support graph*), the set of *configurations* on  $G$  and an *evolution rule* on this set of configurations. Here, a configuration  $c$  on  $G$  is a map from the set  $V(G)$  of vertices of  $G$  to non-negative integers. For each vertex  $v$ , the integer  $c(v)$  is regarded as the number of chips stored in  $v$ . In a configuration  $c$ , vertex  $v$  is *firable* if  $v$  has at least one outgoing edge and  $c(v)$  is not smaller than the out-degree of  $v$ . The *evolution rule* is defined as follows. When  $v$  is firable in  $c$ ,  $c$  can be transformed into another configuration  $c'$  by moving one chip stored in  $v$  along each outgoing edge of  $v$ . We call this process *firing*  $v$  and write  $c \xrightarrow{v} c'$ . An *execution* is a sequence of firing and is often written in the form  $c_0 \xrightarrow{v_1} c_1 \xrightarrow{v_2} c_2 \cdots \rightarrow c_{k-1} \xrightarrow{v_k} c_k$ . The set of configurations which can be obtained from  $c$  by a sequence of firing is called *configuration space*, and denoted by  $CFG(G, c)$ .

A CFG begins with an initial configuration  $\mathcal{O}$ . It can be played forever or reaches a unique fixed point where no firing is possible [10]. When the game reaches a unique fixed point,  $CFG(G, \mathcal{O})$  is an *upper locally distributive lattice* with the order defined by setting  $c_1 \leq c_2$  if  $c_1$  can be transformed into  $c_2$  by a (possibly empty) sequence of firing [14]. A CFG is *simple* if each vertex is fired at most once during any execution to reach its fixed point. Two CFGs are *equivalent* if their generated lattices are isomorphic.

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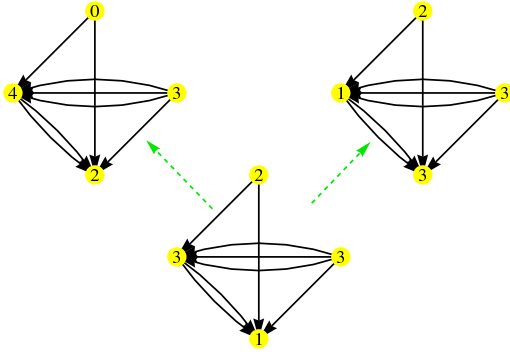


Figure 1: The number at each vertex indicates the number of chips stored there. The configuration at the bottom of the figure can be transformed into two new configurations since it has two firable vertices

Let  $L(CFG)$  denote the class of lattices generated by CFGs. It is a well-known result that  $L(CFG) \subseteq ULD$  [8], where  $ULD$  stands for the class of upper locally distributive lattices. However, a precise structure of  $L(CFG)$  is unknown so far. Our aim is to find a condition to completely characterize this class of lattices. The results in [8] do not say much about the structure of  $L(CFG)$ . One of the most important discovery in our study is finding out a strong connection between the objects which seem to be far from each other. These objects are meet-irreducibles, simple CFGs, firing vertices of a CFG and systems of linear inequalities. In particular, we establish a one-to-one correspondence between the firing vertices of a simple CFG and the meet-irreducibles of the lattice generated by this CFG. Using this correspondence, we achieve a full characterization of  $L(CFG)$  which is practical and can be used to construct an algorithm for the determination of the class of lattices generated by CFGs.

In addition, our method can be applied to other Chip Firing Game models such as the Abelian Sandpile model [15, 1, 6] which has been extensively studied in recent years. This model is a restriction of CFGs on undirected graphs. In [15], the author studied the class of lattices generated by ASMs, denoted by  $L(ASM)$ , and showed that this class of lattices is strictly included in  $L(CFG)$  and completely contains the class of distributive lattices ( $D$ ). A full characterization for this class would strengthen our result. To get the full characterization for  $L(CFG)$ , we use the important result presented in [8], that is a CFG is always equivalent to a simple CFG. A difficulty of getting a full characterization for  $L(ASM)$  is that we do not know whether similar assertion holds for ASM, *i.e.* whether an ASM is equivalent to a simple ASM, therefore the argument in [8] does not seem to be transferable to ASM. However, we overcome this difficulty by constructing a generalized correspondence between the firing vertices in a relation with their times of firing of a CFG and the meet-irreducibles of the lattice generated by this CFG. Finally, we come up with a full characterization for  $L(ASM)$ . Using the algorithm we built, we have found a lattice in  $L(CFG) \setminus L(ASM)$  which is smaller than the one shown in [15].

In [15], to prove  $D \subseteq L(ASM)$ , the author studied simple CFGs on directed acyclic graphs (DAGs) and showed that such a CFG is equivalent to a CFG on an undirected graph. It is natural to study CFGs on DAGs which are not necessary to be simple. Again our method is applicable to this model and we show that any CFG on a DAG is equivalent to a simple CFG on a DAG. As a corollary, the class of lattices generated by CFGs on DAGs is strictly included in  $L(ASM)$ . We also give a full characterization for the class of lattices generated by this model.

Section 2 gives some preliminary definitions, notations and results on lattice and Chip Firing game. Sections 3,4 and 5 are devoted to giving full characterizations on three classes of lattices generated by CFGs on general graphs, undirected graphs and directed acyclic graphs respectively. In the conclusion, we give some open projects which are in our interest for coming works.

## 2. Preliminary definitions and known results

### 2.1. Definitions and notations

Let  $L = (X, \leq)$  be a finite partial order ( $X$  is equipped with a binary relation  $\leq$  which is transitive, reflexive and antisymmetric). For  $x, y \in X$ ,  $y$  is an *upper cover* of  $x$  if  $x < y$  and for every  $z \in X$ ,  $x \leq z \leq y$  implies that  $z = x$  or  $z = y$ . If  $y$  is an upper cover of  $x$  then  $x$  is a *lower cover* of  $y$ , and then we write  $x < y$ . A finite partial order is often presented by a Hasse diagram in which for each cover  $x < y$  of  $L$ , there is a curve that goes upward from  $x$  to  $y$ .  $L$  is a *lattice* if any two elements of  $L$  have a least upper bound (*join*) and a greatest lower bound (*meet*). When  $L$  is lattice, we have the following notations and denitions

- $\mathbf{0}, \mathbf{1}$  denote the minimum and the maximum of  $L$ .
- for every  $x, y \in X$ ,  $x \wedge y, x \vee y$  denote the join and the meet of  $x, y$  respectively.
- for  $x \in X$ ,  $x$  is a *meet-irreducible* if it has only one upper cover.  $x$  is a *join-irreducible* if  $x$  has only one lower cover.  $M$  and  $J$  denote the collections of the meet-irreducibles and the join-irreducibles of  $L$  respectively. Let  $M_x, J_x$  be given by  $M_x = \{m \in M : x \leq m\}$  and  $J_x = \{j \in J : j \leq x\}$ . For  $j \in J, m \in M$ , if  $j$  is a minimal element in  $X \setminus \{x \in X : x \leq m\}$  then we write  $j \downarrow m$ . If  $m$  is a maximal element in  $X \setminus \{x \in X : j \leq x\}$  then we write  $j \uparrow m$ , and  $j \uparrow m$  if  $j \downarrow m$  and  $j \uparrow m$ .
- $L$  is a *distributive lattice* if it satisfies one of the following equivalent conditions
  1. for every  $x, y, z \in X$ , we have  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ .
  2. for every  $x, y, z \in X$ , we have  $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ .

For a finite set  $A$ ,  $(2^A, \subseteq)$  is a distributive lattice. A lattice generated in this way is called *hypercube*.

- for  $x, y \in X$  satisfying  $x \leq y$ ,  $[x, y]$  stands for set  $\{z \in X : x \leq z \leq y\}$ . If  $x \neq \mathbf{1}$ ,  $x^+$  denotes the join of all upper covers of  $x$ . Note that if  $x$  is a meet-irreducible then  $x^+$  is the unique upper cover of  $x$ . If  $x \neq \mathbf{0}$ ,  $x^-$  denotes the meet of all lower covers of  $x$ . If  $x$  is a join-irreducible then  $x^-$  is the unique lower cover of  $x$ .  $L$  is an *upper locally distributive (ULD) lattice* if for every  $x \in X$ ,  $x \neq \mathbf{1}$  implies the sublattice induced by  $[x, x^+]$  is a hypercube. By dual notion,  $L$  is a *lower locally distributive (LLD) lattice* if for every  $x \in X$ ,  $x \neq \mathbf{0}$  implies that the sublattice induced by  $[x^-, x]$  is a hypercube.

Let  $G$  be a directed multi-graph. For  $v_1, v_2 \in V$ ,  $E(v_1, v_2)$  denotes the number of edges from  $v_1$  to  $v_2$ .  $E(v_1, v_1)$  is the number of loops at  $v_1$ . For  $v \in V$ , the out-degree of  $v$ , denoted by  $deg^+(v)$ , is defined by  $deg^+(v) = \sum_{v' \in V} E(v, v')$  and the in-degree of  $v$ , denoted by  $deg^-(v)$ , is defined by  $deg^-(v) = \sum_{v' \in V} E(v', v)$ . A vertex  $v$  of  $G$  is called *sink* if it has no outgoing edge, i.e.  $deg^+(v) = E(v, v)$ . A subset  $C$  of  $V(G)$  is a *closed component* if  $|C| \geq 2$ ,  $C$  is a strongly connected component and there is no edge going from  $C$  to a vertex outside of  $C$ . A CFG, which is defined on a graph having no closed component, always reaches a unique fixed point, moreover  $CFG(G, \mathcal{O})$  is a ULD lattice [5, 14]. If  $CFG(G, \mathcal{O})$  reaches a unique fixed point and  $CFG(G, \mathcal{O})$  is isomorphic to a ULD lattice  $L$ , we say  $CFG(G, \mathcal{O})$  generates  $L$ . Then we can identify the configurations of  $CFG(G, \mathcal{O})$  with the elements of  $L$ .

### 2.2. Known results

**Theorem 1.** [4] *A lattice is distributive if and only if it is isomorphic to the lattice of the ideals of the order induced by its meet-irreducibles.*

**Lemma 1.** [7] A lattice  $L = (X, \leq)$  is upper locally distributive if and only if for any  $x, y \in X$ ,

$$x < y \Leftrightarrow M_y \subset M_x \text{ and } |M_x \setminus M_y| = 1$$

**Lemma 2.** [14] In a CFG reaching a unique fixed point, if two sequences of firing are starting at the same configuration and leading to the same configuration then for every  $v \in V(G)$ , the number of times  $v$  fired in each sequences are the same, where  $G$  is the support on which the game define.

In a  $CFG(G, \mathcal{O})$  reaching a unique fixed point, for each  $c$  being a configuration in the configuration space  $\mathcal{C}$  of the game, the *shot-vector* of  $c$ , denoted by  $sh_c$ , assigns each vertex  $v$  of  $G$  to the number of times  $v$  fired in any execution from the initial configuration to  $c$ . Thus  $sh_c$  is a map from  $V(G)$  to  $\mathbb{N}$ . It follows from the above lemma that shot-vector of  $c$  is well-defined. For  $c_1, c_2 \in \mathcal{C}$ ,  $sh_{c_1} \leq sh_{c_2}$  if for every  $v \in V(G)$ ,  $sh_{c_1}(v) \leq sh_{c_2}(v)$ . It is known that  $sh_{c_1} \leq sh_{c_2}$  if and only if  $c_1$  can be transformed into  $c_2$  by a sequence of firing [14].

Throughout the coming sections, we always work with a general finite ULD lattice  $L = (X, \leq)$ . Recall that  $M, J$  denote the collections of the meet-irreducibles and the join-irreducibles of  $L$  respectively. On  $L$ , we define the map  $m : \{(x, y) : x \in X, y \in X \text{ and } x < y\} \rightarrow M$  by for every  $x < y$  in  $L$ ,  $m(x, y)$  is the unique element in  $M_x \setminus M_y$ . All graphs are supposed to be directed multi-graphs. In a CFG, if configuration  $c$  can be transformed into  $c'$  by firing some vertex in the support graph then we denote this unique vertex by  $\vartheta(c, c')$ . All CFGs, which are considered in this paper, are assumed to be reaching a fixed point. To denote a CFG, a configuration space and a lattice generated by a CFG, we will use the common notation  $CFG(G, \mathcal{O})$  since all of them are defined completely by  $G$  and  $\mathcal{O}$ . This notation will cause no confusion.

### 3. A necessary and sufficient condition for $L(CFG)$

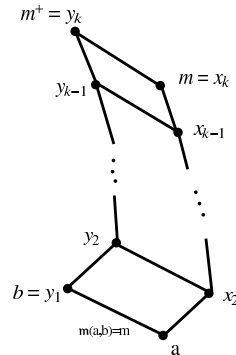
Given a ULD lattice  $L$ , is  $L$  in  $L(CFG)$ ? This question was asked in [8]. Up to this point, there exists no full characterisation for  $L(CFG)$  which allows to induce an algorithm for this computational problem. In this section, we solve the problem by giving a necessary and sufficient condition for  $L(CFG)$ . We recall an important result in [8]

**Theorem 2.** [8] Any CFG that reaches a unique fixed point is equivalent to a simple CFG

So from now until the end of this section, we assume that all CFGs are simple.

**Lemma 3.** Let  $a, b$  be two elements of  $L$  such that  $a < b$ . Let  $m$  denote  $m(a, b)$ . Then for any chain  $a = x_1 < x_2 < \dots < x_k = m$  in  $L$ , there exists a chain  $b = y_1 < y_2 < \dots < y_k = m^+$  in  $L$  such that  $x_i < y_i$  for every  $1 \leq i \leq k$ . Moreover  $m(x_i, y_i) = m$  for every  $1 \leq i \leq k$ .

*Proof.*



It's clear that  $x_2 \neq y_1$ . Since  $L$  is a ULD lattice, there exists a unique  $y_2$  such that  $y_1 < y_2$  and  $x_2 < y_2$ . It follows easily that  $\mathfrak{m}(x_1, y_1) = \mathfrak{m}(x_2, y_2) = m$ . If  $k = 2$  then  $y_2 = m^+$ . Otherwise repeat the previous argument starting with  $x_2, y_2$  until the index reaches  $k$ . We obtain the sequence  $b = y_1 < y_2 < \dots < y_k = m^+$  which has the desired property  $\square$

**Lemma 4.** *Let  $L$  be a ULD lattice generated by  $CFG(G, \mathcal{O})$  and let  $\mathcal{V}$  denote the set of vertices which are fired in  $CFG(G, \mathcal{O})$ . For each  $c \in CFG(G, \mathcal{O})$ ,  $\vartheta(c)$  denotes the set of vertices which are fired to obtain  $c$ . Then*

1. *The map  $\kappa : M \rightarrow \mathcal{V}$  determined by  $\forall m \in M, \kappa(m) = \vartheta(c, c')$ , where  $c, c'$  are two elements in  $L$  such that  $c < c'$  and  $\mathfrak{m}(c, c') = m$ , is well-defined. Furthermore  $\kappa$  is a bijection.*
2. *For every  $c \in CFG(G, \mathcal{O})$ ,  $\vartheta(c) = \kappa(M \setminus M_c)$ .*

*Proof.*

1. It is clear that  $\kappa$  is defined on whole  $M$  since for every  $m \in M$ ,  $\mathfrak{m}(m, m^+) = m$ . To prove  $\kappa$  is well-defined, it suffices to show that for each  $m \in M$ , if  $\mathfrak{m}(a, b) = m$  then  $\vartheta(a, b) = \vartheta(m, m^+)$ . Let  $a = x_1 < x_2 < \dots < x_k = m$ . By Lemma 3, there exists  $b = y_1 < y_2 < \dots < y_k = m^+$  such that for every  $1 \leq i \leq k$ , we have  $x_i < y_i$ . Therefore  $\vartheta(a, b) = \vartheta(x_1, y_1) = \vartheta(x_2, y_2) = \dots = \vartheta(x_k, y_k) = \vartheta(m, m^+)$ .

It's clear that  $\kappa$  is surjective. To prove  $\kappa$  is a bijective, it suffices to show that  $|M| = |\mathcal{V}|$ . Let  $\mathcal{O} = \mathbf{0} = c_0 \xrightarrow{v_1} c_1 \xrightarrow{v_2} c_2 \xrightarrow{v_3} \dots \rightarrow c_{N-1} \xrightarrow{v_N} c_N = \mathbf{1}$  be an execution to obtain the fixed point. Since  $M_{\mathbf{0}} = M, M_{\mathbf{1}} = \emptyset$  and for every  $0 \leq i \leq N-1$ ,  $|M_{c_i} \setminus M_{c_{i+1}}| = 1$ , it follows that  $N = |M|$ , therefore  $|\mathcal{V}| = |M|$

2. Let  $\mathbf{0} = d_0 \xrightarrow{v_1} d_1 \xrightarrow{v_2} d_2 \xrightarrow{v_3} \dots \xrightarrow{v_k} d_k = c \xrightarrow{v_{k+1}} d_{k+1} \rightarrow \dots \xrightarrow{v_{N-1}} d_{N-1} \xrightarrow{v_N} d_N = \mathbf{1}$  be an execution to obtain the fixed point. It's clear that  $\vartheta(c) = \{v_1, v_2, \dots, v_k\}$ , therefore  $\vartheta(c) = \{v_1, v_2, \dots, v_N\} \setminus \{v_{k+1}, v_{k+2}, \dots, v_N\}$ . By the definition of  $\kappa$ , we have  $\kappa(M) = \{v_1, v_2, \dots, v_N\}$  and  $\{v_{k+1}, v_{k+2}, \dots, v_N\} = \{\kappa(\mathfrak{m}(d_i, d_{i+1})) : k \leq i \leq N-1\} = \kappa(\{\mathfrak{m}(d_i, d_{i+1}) : k \leq i \leq N-1\}) = \kappa(\bigcup_{k \leq i \leq N-1} M_{d_i} \setminus M_{d_{i+1}}) = \kappa(M_{d_k} \setminus M_{d_N}) = \kappa(M_{d_k}) = \kappa(M_c)$ . It follows easily that  $\vartheta(c) = \kappa(M) \setminus \kappa(M_c) = \kappa(M \setminus M_c)$  since  $\kappa$  is bijective  $\square$

The lemma means that if  $L$  is generated by a CFG then each meet irreducible of  $L$  can be considered as a vertex of its support graph. It's an important point to set up a full characterization for  $L(CFG)$ . For better understanding, we give an example for this correspondence. The CFG which is defined on the support graph and the initial configuration shown in Figure 2a and Figure 2b generates the lattice represented in Figure 3a.

In Figure 3a, each  $c_i < c_j$  is labeled by the vertex which is fired in  $c_i$  to obtain  $c_j$ . The lattice in Figure 4a is the same as one in Figure 3a but each  $c_i < c_j$  is labeled by  $\mathfrak{m}(c_i, c_j)$ . Figure 3b shows the lattice in the way each configuration is presented by the set of vertices which are fired to obtain this configuration. In Figure 4b, each configuration  $c$  is presented by  $M \setminus M_c$ . It's easy to see that the labelings in Figure 3a and Figure 4a are the same, the presentations in Figure 3b and Figure 4b are the same too with respect to the correspondence  $\kappa$  defined by  $\kappa(c_6) = v_4, \kappa(c_7) = v_3, \kappa(c_8) = v_2, \kappa(c_9) = v_1$ .

For each  $m \in M$ ,  $\mathfrak{U}_m$  denotes the collection of all minimal elements of  $\{x \in X : \exists y \in X, x < y \text{ and } \mathfrak{m}(x, y) = m\}$  and  $\mathfrak{Q}_m$  denotes the collection of all maximal elements of  $X \setminus \bigcup_{a \in \mathfrak{U}_m} \{x \in X : a \leq x\}$ .

We explain in a few words why we define these notations. Suppose that  $L$  is generated by  $CFG(G, \mathcal{O})$ . For a vertex  $v$  fired in the game, we consider all configurations in  $CFG(G, \mathcal{O})$  which have enough chips

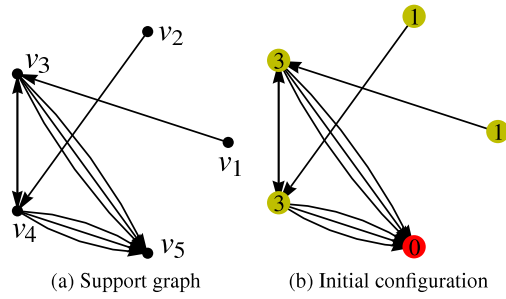


Figure 2: An example of Chip firing game

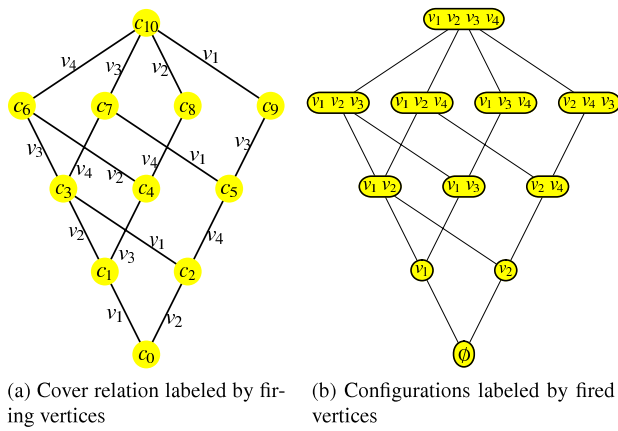


Figure 3: firing-vertex labeling

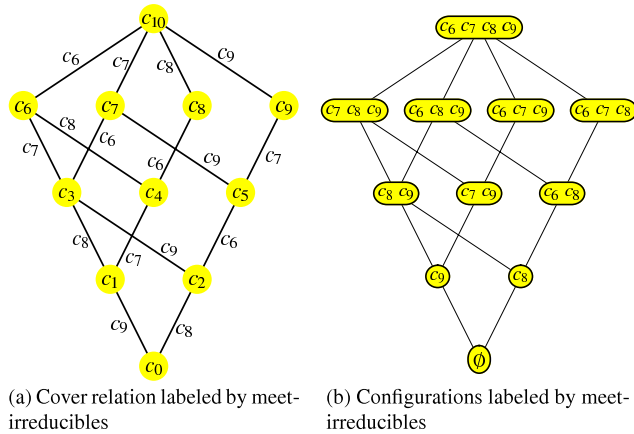


Figure 4: meet-irreducible labeling

stored at  $v$  in order that  $v$  can be fired. If we only care about the firability of  $v$ , we need only to consider the collection  $\mathfrak{U}_v$  of all minimal configurations of these configurations. It is clear that the configurations, which are not greater than equal to any configuration in  $\mathfrak{U}_v$ , do not have enough chips stored at  $v$  in order that  $v$  can be fired. We need only to consider the collection  $\mathfrak{Q}_v$  of all maximal configurations of these configurations to know the firability of  $v$ . Sets  $\mathfrak{U}_v, \mathfrak{Q}_v$  are exactly  $\mathfrak{U}_{\kappa^{-1}(v)}, \mathfrak{Q}_{\kappa^{-1}(v)}$  respectively in  $L$ . Note that  $\mathfrak{U}_m, \mathfrak{Q}_m$  are defined independently no matter existing or not a CFG generating  $L$ .

With the above notations, we create the system of linear inequalities  $\mathcal{E}(m)$  as follows

$$\mathcal{E}(m) = \begin{cases} \left\{ \sum_{x \in M \setminus M_a} e_x < w : a \in \mathfrak{Q}_m \right\} \cup \left\{ w \leq \sum_{x \in M \setminus M_a} e_x : a \in \mathfrak{U}_m \right\} & \text{if } \mathfrak{U}_m \neq \{\mathbf{0}\} \\ \{w \geq 1\} & \text{if } \mathfrak{U}_m = \{\mathbf{0}\} \end{cases}$$

where  $w$  is an added variable. The collection of all variables of  $\mathcal{E}(m)$  is  $\{w\} \cup \{e_x : x \in \bigcup_{a \in \mathfrak{U}_m \cup \mathfrak{Q}_m} (M \setminus M_a)\}$ . It follows from the definitions of  $\mathfrak{U}_m$  and  $\mathfrak{Q}_m$  that if  $e_x$  is in the collection of all variables of  $\mathcal{E}(m)$  then  $x \neq m$ . Note that  $\mathcal{E}(m) = \{w \geq 1\}$  if and only if there exists  $x \in X$  such that  $\mathbf{0} < x$  and  $m(\mathbf{0}, x) = m$ .

**Example 1.** We consider again the lattice presented in Figure 4a. We have  $M = \{c_6, c_7, c_8, c_9\}$ ,  $\mathfrak{U}_{c_8} = \mathfrak{U}_{c_9} = \{c_0\}$ ,  $\mathfrak{U}_{c_6} = \{c_2, c_4\}$ ,  $\mathfrak{U}_{c_7} = \{c_1, c_5\}$ ,  $\mathfrak{Q}_{c_8} = \mathfrak{Q}_{c_9} = \emptyset$ ,  $\mathfrak{Q}_{c_6} = \{c_1\}$ ,  $\mathfrak{Q}_{c_7} = \{c_2\}$ . Then

$$\mathcal{E}(c_8) = \mathcal{E}(c_9) = \{w \geq 1\}$$

$$\mathcal{E}(c_6) = \begin{cases} w \leq e_{c_8} \\ w \leq e_{c_7} + e_{c_9} \\ e_{c_9} < w \end{cases}; \mathcal{E}(c_7) = \begin{cases} w \leq e_{c_9} \\ w \leq e_{c_6} + e_{c_8} \\ e_{c_8} < w \end{cases}$$

**Lemma 5.** *If  $L \in L(\text{CFG})$  then for every  $m \in M$ ,  $\mathcal{E}(m)$  has non-negative integer solutions.*

*Proof.* The lemma clearly holds if  $\mathcal{E}(m) = \{w \geq 1\}$ . If  $\mathcal{E}(m) \neq \{w \geq 1\}$ , there exists a CFG, say  $\text{CFG}(G, \mathcal{O})$ , which generates  $L$ . We can identify the elements of  $L$  with the configurations of  $\text{CFG}(G, \mathcal{O})$ . Without loss of generality, we can assume that all vertices of  $G$  are fired exactly once in the game except for only one vertex  $s$ . Note that  $\text{CFG}(G, \mathcal{O})$  remains unchanged if we remove all out-edges of  $s$ , therefore  $s$  can be considered as the sink of the game.

Let  $f_m : \{e_x : x \in \bigcup_{a \in \mathfrak{Q}_m \cup \mathfrak{U}_m} (M \setminus M_a)\} \cup \{w\} \rightarrow \mathbb{N}$  be the map defined by

$$f_m(y) = \begin{cases} E(\kappa(x), \kappa(m)) & \text{if } y = e_x \text{ for some } x \in \bigcup_{a \in \mathfrak{U}_m \cup \mathfrak{Q}_m} (M \setminus M_a) \\ \text{deg}^+(\kappa(m)) - \mathcal{O}(\kappa(m)) & \text{if } y = w \end{cases}$$

where  $\kappa$  is the map which is defined as in Lemma 4. Note that since  $\mathcal{E}(m) \neq \{w \geq 1\}$ ,  $\kappa(m)$  can not be fired at the beginning of the game, therefore  $\text{deg}^+(\kappa(m)) - \mathcal{O}(\kappa(m)) > 0$ .

We claim that  $f_m$  is a solution of  $\mathcal{E}(m)$ . Indeed, let  $a \in \mathfrak{U}_m$ . By Lemma 4, the set of vertices which are fired to obtain  $a$  is  $\kappa(M \setminus M_a)$ . After firing all vertices in  $\kappa(M \setminus M_a)$ ,  $\kappa(m)$  receives  $\sum_{x \in M \setminus M_a} E(\kappa(x), \kappa(m))$  chips from its neighbors. Since  $\kappa(m)$  is firable in  $a$ , it follows that  $\sum_{x \in M \setminus M_a} f_m(e_x) = \sum_{x \in M \setminus M_a} E(\kappa(x), \kappa(m)) \geq \text{deg}^+(\kappa(m)) - \mathcal{O}(\kappa(m)) = f_m(w)$ . It remains to prove that for  $a' \in \mathfrak{Q}_m$ , we have  $\sum_{x \in M \setminus M_{a'}} f(e_x) < f(w)$ . It follows from the definition of  $\mathfrak{Q}_m$  and from Lemma 4 that  $\kappa(m)$  is not firable in  $a'$ . By a similar argument, we have  $\sum_{x \in M \setminus M_{a'}} f_m(e_x) = \sum_{x \in M \setminus M_{a'}} E(\kappa(x), \kappa(m)) < \text{deg}^+(\kappa(m)) - \mathcal{O}(\kappa(m)) = f_m(w)$   $\square$



**Theorem 3.**  $L$  is in  $L(CFG)$  if and only if for each  $m \in M$ ,  $\mathcal{E}(m)$  has non-negative integer solutions.

*Proof.*  $\Rightarrow$  has been proved by Lemma 5. It remains to show that  $\Leftarrow$  is also true. We are going to construct a graph  $G$  and an initial configuration  $\mathcal{O}$  so that the game is simple and  $CFG(G, \mathcal{O})$  is isomorphic to  $L$ .

The set of vertices of  $G$  is  $M \cup \{s\}$ , where  $s$  is distinct from  $M$  and will play a role as the sink of  $G$ . The edges of  $G$  are constructed as follows. For each  $m \in M$ , let  $f_m : U_m \rightarrow \mathbb{N}$  be a solution of  $\mathcal{E}(m)$ , where  $U_m$  is the collection of all variables in  $\mathcal{E}(m)$ . We set  $E(m, s) = f_m(w) + \sum_{v \in M \text{ and } e_v \in U_m \setminus \{w\}} f_m(e_v)$  and for each  $v \in M$  satisfying  $e_v \in U_m \setminus \{w\}$ ,  $E(v, m) = f_m(e_v)$ . Note that with this construction,  $G$  has no loop. Furthermore  $G$  has no closed component since each vertex  $v \neq s$  has at least one edge going from  $v$  to  $s$ , therefore any CFG on  $G$  reaches a fixed point. Next, we construct  $\mathcal{O} : V(G) \rightarrow \mathbb{N}$  as follows

$$\mathcal{O}(v) = \begin{cases} deg^+(v) - f_v(w) & \text{if } v \neq s \text{ and } deg^-(v) \neq 0 \\ deg^+(v) & \text{if } deg^-(v) = 0 \\ 0 & \text{if } v = s \end{cases}$$

We claim that  $CFG(G, \mathcal{O})$  is simple. Indeed, for the sake of contradiction, we suppose that there exists at least one vertex in  $G$  which is fired more than once in an execution, say  $\mathcal{O} = c_0 \xrightarrow{v_1} c_1 \xrightarrow{v_2} c_2 \cdots \xrightarrow{v_{k-1}} c_{k-1} \xrightarrow{v_k} c_k$ , to reach the fixed point of  $CFG(G, \mathcal{O})$ . By the assumption,  $v_1, v_2, \dots, v_k$  are not pairwise distinct. Let  $i$  be the largest index such that  $v_1, v_2, \dots, v_{i-1}$  are pairwise distinct. Vertex  $v_i$ , therefore, is in  $\{v_1, v_2, \dots, v_{i-1}\}$ . We have  $deg^-(v_i) \neq 0$  since  $v_i$  is fired more than once during the execution. To obtain  $c_{i-1}$ , each vertex in  $v_1, v_2, \dots, v_{i-1}$  is fired exactly once, therefore  $c_{i-1}(v_i) \leq c_0(v_i) + deg^-(v_i) - deg^+(v_i)$ . Since  $v_i$  is fireable in  $c_{i-1}$ , it follows that  $deg^+(v_i) \leq c_{i-1}(v_i) \leq c_0(v_i) + deg^-(v_i) - deg^+(v_i) = -f_{v_i}(w) + deg^-(v_i)$ . It follows easily from the construction of  $G$  that  $deg^+(v_i) \geq deg^-(v_i)$ . It is a contradiction.

We claim that for every execution  $\mathcal{O} = c_0 \xrightarrow{v_1} c_1 \xrightarrow{v_2} c_2 \xrightarrow{v_3} \cdots \xrightarrow{v_{k-1}} c_{k-1} \xrightarrow{v_k} c_k$  of  $CFG(G, \mathcal{O})$ , there exists a chain  $\mathbf{0} = d_0 < d_1 < d_2 < \cdots < d_{k-1} < d_k$  in  $L$  such that  $m(d_{i-1}, d_i) = v_i$  for every  $1 \leq i \leq k$ . Note that if the chain exists then it is defined uniquely. We prove the claim by induction on  $k$ . For  $k = 1$ ,  $v_1$  is fireable in  $c_0$ . It follows from the construction of  $G$  and  $\mathcal{O}$  that only the vertices in  $G$  having indegree 0 are fireable in  $\mathcal{O}$ , therefore  $\mathfrak{U}_{v_1} = \{\mathbf{0}\}$ . It implies that there exists  $d_1 \in X$  such that  $d_0 < d_1$  and  $M_{d_0} \setminus M_{d_1} = \{v_1\}$ . The claim holds for  $k = 1$ . For  $k \geq 2$ , let  $\mathbf{0} < d_1 < d_2 < \cdots < d_{k-1}$  be the chain in  $L$  such that  $M_{d_{i-1}} \setminus M_{d_i} = \{v_i\}$  for every  $1 \leq i \leq k-1$ . If  $d_{k-1}$  is not less than or equal to any element in  $\mathfrak{U}_{v_k}$  then there exists  $a \in \mathfrak{Q}_{v_k}$  such that  $d_{k-1} \leq a$ . It follows from the definition of  $\mathcal{E}(v_k)$  that  $\sum_{i=1}^{k-1} f_{v_k}(e_{v_i}) \leq \sum_{x \in M \setminus M_a} f_{v_k}(e_x) < f_{v_k}(w)$ . It implies that after  $v_1, v_2, \dots, v_{k-1}$  have been fired,  $v_k$  receives less than  $f_{v_k}(w)$  chips from its neighbors, therefore  $v_k$  is not fireable in  $c_{k-1}$ . It's a contradiction. If there exists  $b \in \mathfrak{U}_{v_k}$  such that  $b \leq d_{k-1}$  then there exists  $b' \in X$  such that  $b < b'$  and  $M_b \setminus M_{b'} = \{v_k\}$ . Let  $d_k = b' \vee d_{k-1}$ . It suffices to show that  $M_{d_{k-1}} \setminus M_{d_k} = \{v_k\}$ . Indeed, we have  $M_{d_k} = M_{b'} \cap M_{d_{k-1}} = (M_b \setminus \{v_k\}) \cap M_{d_{k-1}} = (M_b \cap M_{d_{k-1}}) \setminus \{v_k\} = M_{d_{k-1}} \setminus \{v_k\}$ . Since  $M \setminus M_{d_{k-1}} = \{v_1, v_2, \dots, v_{k-1}\}$ ,  $v_k \in M_{d_{k-1}}$ , therefore  $M_{d_{k-1}} \setminus M_{d_k} = \{v_k\}$ .

Our next claim is that for any chain  $\mathbf{0} = d_0 < d_1 < d_2 < \cdots < d_{k-1} < d_k$  in  $L$ , there exists an execution  $\mathcal{O} = c_0 \xrightarrow{v_1} c_1 \xrightarrow{v_2} c_2 \xrightarrow{v_3} \cdots \xrightarrow{v_{k-1}} c_{k-1} \xrightarrow{v_k} c_k$  in  $CFG(G, \mathcal{O})$ , where  $v_i = m(d_{i-1}, d_i)$  for every  $1 \leq i \leq k$ . We prove the claim by induction on  $k$ . For  $k = 1$ , we have  $\mathfrak{U}_{v_1} = \{\mathbf{0}\}$ . It follows easily that  $deg^-(v_1) = 0$ , therefore  $v_1$  is fireable in  $\mathcal{O}$ . By firing  $v_1$  in  $c_0$ , we obtain  $c_0 \xrightarrow{v_1} c_1$ . The claim holds for  $k = 1$ . For  $k \geq 2$ , let  $\mathcal{O} = c_0 \xrightarrow{v_1} c_1 \xrightarrow{v_2} c_2 \xrightarrow{v_3} \cdots \xrightarrow{v_{k-1}} c_{k-1}$  be an execution in the game such that  $\{v_i\} = M_{d_{i-1}} \setminus M_{d_i}$  for every  $1 \leq i \leq k-1$ . There is  $a \in \mathfrak{U}_{v_k}$  such that  $a \leq d_{k-1}$ , therefore  $M \setminus M_a \subseteq \{v_1, v_2, \dots, v_{k-1}\}$ .  $v_k$  receives at least  $\sum_{x \in M \setminus M_a} f_{v_k}(e_x)$  chips from its neighbors after all vertices  $v_1, v_2, \dots, v_{k-1}$  have been fired.  $v_k$  is fireable in  $c_{k-1}$

since  $\sum_{x \in M \setminus M_a} f_{v_k}(e_x) \geq f_{v_k}(w)$ . The claim holds.

It follows easily from the above claims that  $CFG(G, O)$  and  $L$  are isomorphic  $\square$

Next, we will establish a relation between  $\mathfrak{U}_m$  and the join-irreducibles of  $L$ . The main result (Theorem 4.1 in [8]) will follow easily from this relation.

**Proposition 1.** *For each meet-irreducible  $m$  of  $L$ ,  $\mathfrak{U}_m = \{j^- : j \in J \text{ and } j \downarrow m\}$*

*Proof.* For each  $m \in M$ , let  $\mathcal{F}_m$  be given by  $\mathcal{F}_m = \{x \in X : \exists y \in X, x < y \text{ and } m(x, y) = m\}$ . Let  $A$  denote  $\{j^- : j \in J \text{ and } j \downarrow m\}$ . First, we show that  $A \subseteq \mathfrak{U}_m$ . To this end, we prove that  $j^- \in \mathfrak{U}_m$  for every  $j \in J$  satisfying  $j \downarrow m$ . Since  $j \not\leq m$  and  $j^- \leq m$ , we have  $m \notin M_j$  and  $m \in M_{j^-}$ , therefore  $m \in M_{j^-} \setminus M_j$ . Since  $|M_{j^-} \setminus M_j| = 1$ , it follows that  $M_{j^-} \setminus M_j = \{m\}$ , hence  $j^- \in \mathcal{F}_m$ . It remains to prove that  $j^-$  is a minimal element of  $\mathcal{F}_m$ . For a contradiction, we suppose that there exists  $a < b$  in  $L$  such that  $m(a, b) = m$  and  $a < j^-$ . It follows easily that  $b < j$ , therefore there is a chain  $b = d_1 < d_2 < \dots < d_k = j$  in  $L$  of length  $\geq 1$ . We have  $m(d_{k-1}, j) \neq m$  since  $m(a, b) = m$ , therefore  $d_{k-1} \neq j^-$ . It contradicts the fact that  $j$  is a join-irreducible.

We are left with showing that  $\mathfrak{U}_m \subseteq A$ . Let  $a \in \mathfrak{U}_m$ . There is a unique element  $b$  in  $L$  such that  $a < b$  and  $m(a, b) = m$ . It suffices to show that  $b \in J$ . For a contradiction, we suppose that  $b \notin J$ . Then there exists  $c \in X$  such that  $c < b$  and  $c \neq a$ . Let  $d$  denote the infimum of  $a$  and  $c$ . There exists  $a' \in L$  such that  $d < a'$  and  $a' \leq c$ . Since  $a \in \mathfrak{U}_m$ , we have  $m(d, a') \neq m$ , therefore  $m \in M_{a'}$ . It follows from  $a' \leq b$  that  $M_a = M_b \cup \{m\} \subseteq M_{a'} \cup \{m\} = M_{a'}$ , hence  $a' \leq a$ . It contradicts the fact that  $d$  is the infimum of  $a$  and  $c$   $\square$

**Corollary 1.** *If  $L$  is a distributive lattice then for every meet-irreducible  $m$  of  $L$ , we have  $|\mathfrak{U}_m| = 1$ .*

*Proof.* For each  $m \in M$ , we define  $m_\downarrow = \{j \in J : j \downarrow m\}$ . Note that  $m_\downarrow \neq \emptyset$ . For every  $m_1, m_2 \in M$ ,  $m_1 \neq m_2$  implies that  $m_{1\downarrow} \cap m_{2\downarrow} = \emptyset$  since if  $j \in m_{1\downarrow}$  then  $m(j^-, j) = m_1$ . In a distributive lattice, the cardinality of the meet-irreducibles is equal to the cardinality of the join-irreducibles, i.e  $|M| = |J|$ . It follows easily that for every  $m \in M$ ,  $|m_\downarrow| = 1$ , therefore  $|\mathfrak{U}_m| = 1$  by Proposition 1  $\square$

**Proposition 2.** *If  $L$  is a distributive lattice then for each  $m \in M$ ,  $\mathcal{E}(m)$  has non-negative integer solutions.*

*Proof.* It follows from Corollary 1 that  $|\mathfrak{U}_m| = 1$ . If  $\mathfrak{U}_m = \{\mathbf{0}\}$  then  $\mathcal{E}(m) = \{w \geq 1\}$ , therefore the proposition holds. If  $\mathfrak{U}_m \neq \{\mathbf{0}\}$ , let  $u$  denote the unique element in  $\mathfrak{U}_m$ .  $\mathcal{E}(m)$  now becomes

$$\mathcal{E}(m) = \left\{ \sum_{x \in M \setminus M_a} e_x < w : a \in \mathfrak{Q}_m \right\} \cup \left\{ w \leq \sum_{x \in M \setminus M_u} e_x \right\}$$

The collection  $U_m$  of all variables in  $\mathcal{E}(m)$  is  $\{w\} \cup \{e_x : x \in (M \setminus M_u) \cup \bigcup_{a \in \mathfrak{Q}_m} (M \setminus M_a)\}$ . Let  $f : U_m \rightarrow \mathbb{N}$  be given by

$$f(y) = \begin{cases} 1 & \text{if } y = e_x \text{ for some } x \in M \setminus M_u \\ |M \setminus M_u| & \text{if } y = w \\ 0 & \text{otherwise} \end{cases}$$

We claim that  $f$  is a solution of  $\mathcal{E}(m)$ . By the definition of  $f$ , it is straightforward to verify that  $f(w) \leq \sum_{x \in M \setminus M_u} f(e_x)$ . It remains to show that for every  $a \in \mathfrak{Q}_m$ ,  $\sum_{x \in M \setminus M_a} f(e_x) < f(w)$ . Indeed,  $\sum_{x \in M \setminus M_a} f(e_x) = |(M \setminus M_a) \cap (M \setminus M_u)|$  follows from the definition of  $f$ . Since  $u \not\leq a$ , we have  $(M \setminus M_a) \cap (M \setminus M_u) \subsetneq M \setminus M_u$ , therefore  $\sum_{x \in M \setminus M_a} f(e_x) < |M \setminus M_u| = f(w)$   $\square$

We derive easily the following corollary from Theorem 3 and Proposition 2

**Corollary 2.** [8] *Every distributive lattice is in  $L(CFG)$ .*

We give an algorithm to construct a CFG which allows to generate a given ULD lattice if such CFG exists. In practice,  $L$  can be input as an acyclic graph with the edges induced from the cover relation of  $L$ , in other word  $(x, y) \in E(L)$  if and only if  $x < y$  in  $L$ .

We use the following facts for building the algorithm and presenting the complexity of the algorithm

1. For each  $m \in M$ ,  $\mathfrak{U}_m$  and  $\mathfrak{Q}_m$  can be computed in  $O(|E(L)|)$  by using the search algorithms.
2. For each  $m \in M$ ,  $\mathcal{E}(m)$  has non-negative integer solutions if and only if  $\mathcal{E}'(m)$ , which is defined by if  $\mathfrak{U}_m \neq \{\mathbf{0}\}$  then  $\mathcal{E}'(m) = \{1 \leq w - \sum_{x \in M \setminus M_a} e_x : a \in \mathfrak{Q}_m\} \cup \{w \leq \sum_{x \in M \setminus M_a} e_x : a \in \mathfrak{U}_m\}$ , otherwise  $\mathcal{E}'(m) = \{w \geq 1\}$ , has non-negative solutions on  $\mathbb{R}$ . The claim clearly holds for  $\mathcal{E}'(m) = \{w \geq 1\}$ . If  $\mathcal{E}'(m) \neq \{w \geq 1\}$ , let  $f'$  be a non-negative solution of  $\mathcal{E}'(m)$ . The map  $f : \{w\} \cup \{e_x : x \in \bigcup_{a \in \mathfrak{Q}_m \cup \mathfrak{U}_m} (M \setminus M_a)\} \rightarrow \mathbb{N}$  defined by  $f(e_x) = \lfloor 2Nf'(e_x) \rfloor$  for every  $x \in \bigcup_{a \in \mathfrak{Q}_m \cup \mathfrak{U}_m} (M \setminus M_a)$  and  $f(w) = \min\{\sum_{x \in M \setminus M_a} f(e_x) : a \in \mathfrak{U}_m\}$ , where  $N = |\bigcup_{a \in \mathfrak{Q}_m \cup \mathfrak{U}_m} (M \setminus M_a)|$ , is a non-negative integer solution of  $\mathcal{E}(m)$ . It is left to the reader to verify that  $f$  is a non-negative integer solution of  $\mathcal{E}(m)$ . Thus we can use the algorithms for linear programming to find a non-negative integer solution of  $\mathcal{E}(m)$  if such a solution exists.
3. For each  $m \in M$ , the number of bits, which are input to the algorithms for linear programming to find a non-negative integer solution of  $\mathcal{E}(m)$ , is  $O(|m_{\downarrow}| \times |X|)$ . In practice, the number of bits is much smaller than  $|m_{\downarrow}| \times |X|$ .

We obtain an algorithm for the CFG reconstruction

**Input** : A ULD lattice  $L$  which is input as a acyclic graph with the edges defined by the cover relation

**Output**: **Yes** if  $L$  is in  $L(CFG)$ , **No** otherwise. In the case  $L$  is in  $L(CFG)$ , give a support graph  $G$  and an initial configuration  $O$  on  $G$  so that  $CFG(G, O)$  is isomorphic to  $L$

$V(G) := M \cup \{s\};$   
 $E(G) := \emptyset;$

**for**  $m \in M$  **do**  
    Construct  $\mathcal{E}(m)$  ;  
    **if**  $\mathcal{E}(m)$  has no non-negative integer solutions **then Reject**;  
    **else**  
        Let  $f_m$  be a non-negative integer solution of  $\mathcal{E}(m)$ ;  
        Let  $U_m$  be the collection of all variables in  $\mathcal{E}(m)$ ;  
        **for**  $e_x \in U_m \setminus \{w\}$  **do**  
            Add  $f_m(e_x)$  edges  $(x, m)$  to  $G$   
        **end**  
        Add  $f_m(w) + \sum_{e_x \in U_m \setminus \{w\}} f_m(e_x)$  edges  $(m, s)$  to  $G$   
    **end**  
**end**

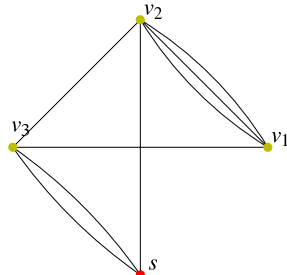
Construct the initial configuration  $O$  by

$$O(v) := \begin{cases} deg^+(v) & \text{if } deg^-(v) = 0 \\ deg^+(v) - f_v(w) & \text{if } deg^-(v) \neq 0 \text{ and } v \neq s \\ 0 & \text{if } v = s \end{cases}$$

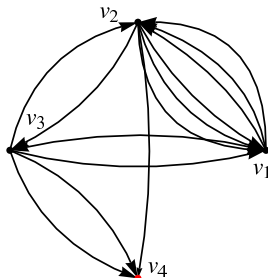
By using Karmarkar's algorithm [13], the runtime of the algorithm is  $O(|M|^{3.5} \times |J|^2 \times |X|^2 \times \log|X| \times \log(\log|X|))$ .

**4. A necessary and sufficient condition for  $L(ASM)$**

Abelian Sandpile model is the CFG model which is defined on connected undirected graphs [1]. In this model, the support graph is undirected and it has a distinguished vertex which is called *sink* and never fires in the game even if it has enough chips. If we replace each undirected edge  $(v_1, v_2)$  in the support graph by two directed edges  $(v_1, v_2)$  and  $(v_2, v_1)$  and remove all out-edges of the sink then we obtain an CFG on directed graph which has the same behavior as the old one. For example, a CFG defined on the following undirected graph with sink  $s$



is the same as one which is defined on the following graph



and the initial configuration is the same as the old one. Thus a *ASM* can be regarded as a CFG on a directed multi-graph. We give an alternative definition of *ASM* on directed multi-graphs as follows. A  $CFG(G, \mathcal{O})$ , where  $G$  is a directed multi-graph, is a *ASM* if  $G$  is connected,  $G$  has only one sink  $s$  and for any two distinct vertices  $v_1, v_2$  of  $G$ , which are distinct from the sink, we have  $E(v_1, v_2) = E(v_2, v_1)$ . Therefore in this model, we will continue to work on directed multi-graphs. The lattice structure of this model was studied in [15]. The authors proved that the class of lattices induced by *ASMs* is strictly included in  $L(CFG)$ . In this section, we give a necessary and sufficient condition for  $L(ASM)$  which allows to build an algorithm in polynomial time for determining whether a given ULD lattice is in  $L(ASM)$ . We also give some other results which concern to this model. The following lemma is a generalization of Lemma 4.

**Lemma 6.** *If  $L$  is generated by  $CFG(G, \mathcal{O})$  then the map  $\kappa : M \rightarrow V(G) \times \mathbb{N}$ , determined by  $\kappa(m) = (\vartheta(c, c'), sh_{\vartheta}(\vartheta(c, c')))$ , where  $c, c'$  are two configurations of  $CFG(G, \mathcal{O})$  such that  $c < c'$  and  $m(c, c') = m$ , is well-defined. Furthermore  $\kappa$  is injective.*

Note that games in Lemma 4 are supposed to be simple, whereas games in the above lemma are general CFGs. The lemma means that if each  $c < c'$  is labeled by the pair of the vertex at which  $c$  is fired to obtain  $c'$  and the number of times this vertex is fired to reach  $c'$  from the initial configuration then the labeling is the same as labeling  $c < c'$  by  $m(c, c')$ . For better understanding this lemma, we give an example. The CFG defined by the support graph  $G$  and the initial configuration  $\mathcal{O}$ , which are presented in Figure 5, generates the lattice shown in Figure 6a and Figure 6b. In Figure 6a, each  $c < c'$  is labeled by the fired

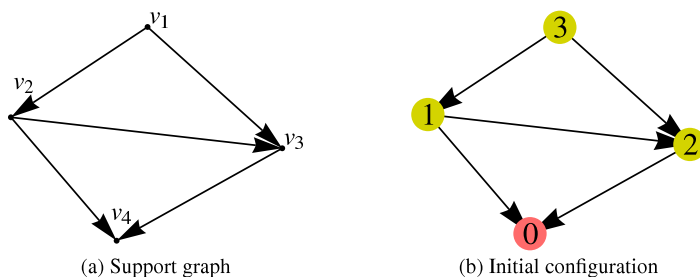


Figure 5: A non-simple CFG

vertex and the number of times this vertex is fired to obtain  $c'$ . Figure 6b shows the lattice in the way each  $c < c'$  is labeled by  $m(c, c')$ . It's obvious that the labelings are the same with respect to the correspondence

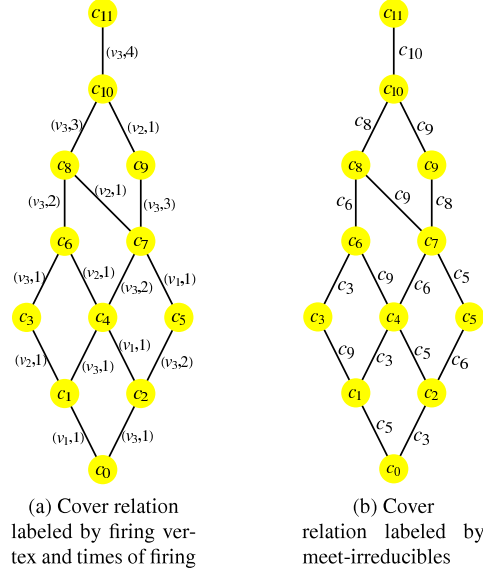


Figure 6: Two ways of labeling

$c_3 \rightarrow (v_3, 1), c_5 \rightarrow (v_1, 1), c_6 \rightarrow (v_3, 2), c_8 \rightarrow (v_3, 3), c_9 \rightarrow (v_2, 1), c_{10} \rightarrow (v_3, 4)$ .

*Proof of Lemma 6.* To prove  $\kappa$  is well-defined, it suffices to show that for  $c, c'$  being two configurations of  $CFG(G, \mathcal{O})$  such that  $m(c, c') = m$ , where  $m \in \mathcal{M}$ , we have  $(\vartheta(c, c'), sh_{c'}(\vartheta(c, c'))) = (\vartheta(m, m^+), sh_{m^+}(\vartheta(m, m^+)))$ . Indeed, let  $c = c_1 < c_2 < c_3 < \dots < c_k = m$  be an execution in  $L$ . By Lemma 3, there exists a chain  $c' = d_1 < d_2 < d_3 < \dots < d_k = m^+$  such that  $c_i < d_i$  for every  $1 \leq i \leq k$ . It is easy to see that  $\vartheta(c, c') = \vartheta(c_1, d_1) = \vartheta(c_2, d_2) = \vartheta(c_3, d_3) = \dots = \vartheta(c_k, d_k) = \vartheta(m, m^+)$ . Let  $v$  denote  $\vartheta(c, c')$ . It remains to prove that  $sh_{c'}(v) = sh_{m^+}(v)$ . For each  $1 \leq i \leq k - 1$ , we have  $v = \vartheta(c_i, d_i) \neq \vartheta(c_i, c_{i+1})$ , therefore  $sh_{d_{i+1}}(v) = 1 + sh_{c_{i+1}}(v) = 1 + sh_{c_i}(v) = sh_{d_i}(v)$ . It implies that  $sh_{c'}(v) = sh_{d_1}(v) = sh_{d_2}(v) = \dots = sh_{d_k}(v) = sh_{m^+}(v)$ .

It follows easily from the definition of  $\kappa$  that  $\kappa$  is a surjection from  $\mathcal{M}$  to  $\bigcup_{v \in V(G)} (\{v\} \times [sh_{\mathbf{1}}(v)])$ . Here  $[n]$  stands for set  $\{1, 2, \dots, n\}$ , by convention,  $[n] = \emptyset$  if  $n \leq 0$ . For  $v \in V(G)$ ,  $|\{v\} \times [sh_{\mathbf{1}}(v)]|$  is the number of times  $v$  is fired in any execution from  $\mathbf{0}$  to  $\mathbf{1}$ , therefore  $|\bigcup_{v \in V(G)} (\{v\} \times [sh_{\mathbf{1}}(v)])|$  is the number of times the vertices are fired to reach  $\mathbf{1}$ . Hence  $|\bigcup_{v \in V(G)} (\{v\} \times [sh_{\mathbf{1}}(v)])|$  is the height of  $L$ . Since  $L$  is a ULD lattice,  $|\mathcal{M}|$  is also the height of  $L$ . Hence  $\kappa$  is injective  $\square$

In the case of directed graphs, we solve the systems  $\mathcal{E}(m)$  of linear inequalities independently to know whether  $L$  is in  $L(CFG)$ . This property no longer holds for the case of undirected graphs since in an undirected graph,  $E(v_1, v_2) = E(v_2, v_1)$  for any two vertices  $v_1, v_2$  distinct from sink. So we construct the systems of linear inequalities as follows.

For each  $\mathcal{E}(m)$ , we define the system of linear inequalities  $\mathfrak{E}(m)$  by replacing each variable  $e_x$  in  $\mathcal{E}(m)$  by  $e_{x,m}$  and  $w$  by  $w_m$ . We give an example for this transformation. Let us consider the lattice shown in Figure 4a. We have

$$\mathcal{E}(c_8) = \mathfrak{E}(c_9) = \{w \geq 1\}$$

$$\mathcal{E}(c_6) = \begin{cases} w \leq e_{c_8} \\ w \leq e_{c_7} + e_{c_9} \\ e_{c_9} < w \end{cases}; \mathcal{E}(c_7) = \begin{cases} w \leq e_{c_9} \\ w \leq e_{c_6} + e_{c_8} \\ e_{c_8} < w \end{cases}$$

then

$$\mathfrak{E}(c_8) = \{w_{c_8} \geq 1\}; \mathfrak{E}(c_9) = \{w_{c_9} \geq 1\}$$

$$\mathfrak{E}(c_6) = \begin{cases} w_{c_6} \leq e_{c_8, c_6} \\ w_{c_6} \leq e_{c_7, c_6} + e_{c_9, c_6} \\ e_{c_9, c_6} < w_{c_6} \end{cases}; \mathfrak{E}(c_7) = \begin{cases} w_{c_7} \leq e_{c_9, c_7} \\ w_{c_7} \leq e_{c_6, c_7} + e_{c_8, c_7} \\ e_{c_8, c_7} < w_{c_7} \end{cases}$$

For each  $m \in M$ ,  $\mathfrak{E}(m)$  is a system of linear inequalities whose variables are a subset of  $\{e_{m_1, m_2} : m_1 \in M, m_2 \in M \text{ and } m_1 \neq m_2\} \cup \{w_m : m \in M\}$ . Let  $U$  denote the set of all variables in  $\bigcup_{m \in M} \mathfrak{E}(m)$ . The system  $\Omega$  of linear inequalities is given by

$$\Omega = \left( \bigcup_{m \in M} \mathfrak{E}(m) \right) \cup \{e_{m_1, m_2} = e_{m_2, m_1} : e_{m_1, m_2} \text{ and } e_{m_2, m_1} \text{ both are in } U\}$$

If  $L$  is generated by a simple CFG, say  $CFG(G, \mathcal{O})$ , then it follows from the correspondence established in Lemma 4 and the construction in Theorem 3 that for  $m_1, m_2 \in M$ ,  $e_{m_1, m_2}$  can be regarded as the number of directed edges from  $v_1$  to  $v_2$  in  $G$ , where  $v_1, v_2$  are the corresponding vertices of  $m_1, m_2$  respectively. For this reason, we come up with the following lemma

**Lemma 7.** *If  $\Omega$  has non-negative integer solutions then  $L \in L(ASM)$ .*

*Sketch of proof.* We construct the graph  $G$  whose the set of vertices is  $M \cup \{s\}$  and the edges are defined in the following manner. Let  $f : U \rightarrow \mathbb{N}$  be a non-negative integer solution of  $\Omega$ . For each two distinct elements  $m_1, m_2 \in M$ , if  $e_{m_1, m_2} \in U$  then there are  $f(e_{m_1, m_2})$  edges connecting  $m_1$  to  $m_2$  in  $G$  and  $f(e_{m_1, m_2})$  edges connecting  $m_2$  to  $m_1$ . If  $e_{m_1, m_2} \notin U$  and  $e_{m_2, m_1} \notin U$  then there is no edge connecting  $m_1$  with  $m_2$  in  $G$ . It follows easily from the definition of  $\Omega$  that  $G$  is well-defined. For each  $m \in M$ , there are  $f(w_m) + \sum_{m' \in M \text{ and } m' \neq m} E(m', m)$  edges connecting  $m$  to  $s$ . The initial configuration  $\mathcal{O} : V(G) \rightarrow \mathbb{N}$  for the game is defined by

$$\mathcal{O}(v) = \begin{cases} deg^+(v) & \text{if } v \in M \text{ and } \mathfrak{U}_v = \{\mathbf{0}\} \\ deg^+(v) - f(w_v) & \text{if } v \in M \text{ and } \mathfrak{U}_v \neq \{\mathbf{0}\} \\ 0 & \text{if } v = s \end{cases}$$

where  $deg^+(v)$  stands for the out-degree of  $v$  in  $G$ . It's clear that  $s$  is the sink of the game. By the same arguments as in the proof of Theorem 3, we can prove easily that the game is simple and generates  $L$ . The details are easy and left to the reader  $\square$

**Example 2.** We consider the system of linear inequalities of Example 1. Then  $\Omega$  is the following system

$$\Omega = \begin{cases} w_{c_8} \geq 1 \\ w_{c_9} \geq 1 \\ w_{c_6} \leq e_{c_8, c_6} \\ w_{c_6} \leq e_{c_7, c_6} + e_{c_9, c_6} \\ e_{c_9, c_6} < w_{c_6} \\ w_{c_7} \leq e_{c_9, c_7} \\ w_{c_7} \leq e_{c_6, c_7} + e_{c_8, c_7} \\ e_{c_8, c_7} < w_{c_7} \\ e_{c_6, c_7} = e_{c_7, c_6} \end{cases}$$

The map  $f : U \rightarrow \mathbb{N}$  defined by

$$f(x) = \begin{cases} 0 & \text{if } x = e_{c_9, c_6} \text{ or } x = e_{c_8, c_7} \\ 1 & \text{otherwise} \end{cases}$$

is a non-negative integer solution of  $\Omega$ . By the construction in the sketch of proof,  $G$  and the initial configuration are presented by Figure 7. Note that in the figure,  $G$  is presented by an undirected graph for a nice

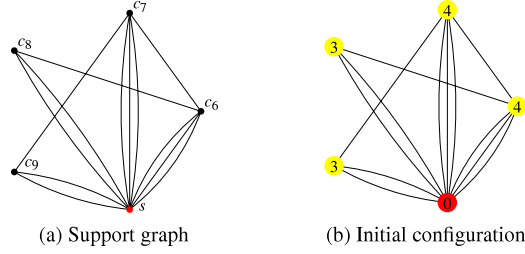


Figure 7: A ASM solution

presentation. The sink of the game is in red. Doing a simple computation on the game, it's straightforward to verify that the lattice generated by  $CFG(G, \mathcal{O})$  is isomorphic to  $L$ .

**Theorem 4.**  $L \in L(ASM)$  if and only if  $\Omega$  has non-negative integer solutions.

*Proof.* The direction  $\Leftarrow$  is already proven by Lemma 7. We are left with proving the direction  $\Rightarrow$ . Let  $CFG(G, \mathcal{O})$  be a ASM and generates  $L$ . We identify the elements of  $L$  with the configurations of  $CFG(G, \mathcal{O})$ .  $s$  denotes the sink of the game. We define  $N = 1 + \sum_{v \in V(G) \text{ and } v \neq s} (sh_1(v) \times deg^+(v))$ , where  $deg^+(v)$  stands for the out-degree of  $v$  in  $G$ .  $\kappa : M \rightarrow V(G) \times \mathbb{N}$  is the injective map which is defined in Lemma 6. For  $m \in M$ ,  $\kappa(m)^{(1)}, \kappa(m)^{(2)}$  denote the first and second components of  $\kappa(m)$  respectively.

We claim that for each  $m \in M$  and each  $a \in \mathcal{Q}_m$ , we have  $|\{(v, n) \in \kappa(M \setminus M_a) : v = \kappa(m)^{(1)}\}| \leq \kappa(m)^{(2)} - 1$ . Indeed, let  $\mathbf{0} = c_0 < c_1 < \dots < c_{k-1} < c_k = a$  be a chain of  $L$ , where  $k$  is a non-negative integer. Note that  $\kappa(M \setminus M_a) = \{\kappa(m(c_i, c_{i+1})) : 0 \leq i \leq k-1\}$ . For a contradiction, we suppose that  $|\{(v, n) \in \kappa(M \setminus M_a) : v = \kappa(m)^{(1)}\}| \geq \kappa(m)^{(2)}$ . It implies that the number of times  $\kappa(m)^{(1)}$  is fired in the execution (chain) is greater than or equal to  $\kappa(m)^{(2)}$ . Hence there is a unique index  $0 \leq j \leq k-1$  such that  $\vartheta(c_j, c_{j+1}) = \kappa(m)^{(1)}$  and  $|\{i : i \leq j \text{ and } \vartheta(c_i, c_{i+1}) = \kappa(m)^{(1)}\}| = \kappa(m)^{(2)}$ . It follows from the definition of  $\kappa$  in Lemma 6 that



$\kappa(m(c_j, c_{j+1})) = \kappa(m)$ . Since  $\kappa$  is injective, it follows that  $m(c_j, c_{j+1}) = m$ . It contradicts the definition of  $\mathfrak{Q}_m$ . The claim holds.

Our next claim is that for each  $m \in M$  and each  $a \in \mathfrak{Q}_m$ , if  $|\{(v, n) \in \kappa(M \setminus M_a) : v = \kappa(m)^{(1)}\}| = \kappa(m)^{(2)} - 1$  then for every  $b \in \mathfrak{U}_m$ , we have

$$\sum_{\substack{x \in M \setminus M_a \\ \kappa(x)^{(1)} \neq \kappa(m)^{(1)}}} E(\kappa(x)^{(1)}, \kappa(m)^{(1)}) < \sum_{\substack{x \in M \setminus M_b \\ \kappa(x)^{(1)} \neq \kappa(m)^{(1)}}} E(\kappa(x)^{(1)}, \kappa(m)^{(1)}) \quad (1)$$

Indeed, the righ-hand side of (1) shows the number of chips vertex  $\kappa(m)^{(1)}$  receives from its neighbors during an execution from  $\mathbf{0}$  to  $b$ . To reach  $b$ ,  $\kappa(m)^{(1)}$  has been fired  $\kappa(m)^{(2)} - 1$  times. It follows that the number of chips stored at  $\kappa(m)^{(1)}$  in  $b$  is

$$\begin{aligned} & \mathcal{O}(\kappa(m)^{(1)}) - (\kappa(m)^{(2)} - 1) \times (\deg^+(\kappa(m)^{(1)}) - E(\kappa(m)^{(1)}, \kappa(m)^{(1)})) + \\ & + \sum_{\substack{x \in M \setminus M_b \\ \kappa(x)^{(1)} \neq \kappa(m)^{(1)}}} E(\kappa(x)^{(1)}, \kappa(m)^{(1)}) \end{aligned} \quad (2)$$

$\kappa(m)^{(1)}$  is firable in  $b$ , therefore (2)  $\geq \deg^+(\kappa(m)^{(1)})$ . By a similar argument, the number of chips stored at  $\kappa(m)^{(1)}$  in  $a$  is

$$\begin{aligned} & \mathcal{O}(\kappa(m)^{(1)}) - (\kappa(m)^{(2)} - 1) \times (\deg^+(\kappa(m)^{(1)}) - E(\kappa(m)^{(1)}, \kappa(m)^{(1)})) + \\ & + \sum_{\substack{x \in M \setminus M_a \\ \kappa(x)^{(1)} \neq \kappa(m)^{(1)}}} E(\kappa(x)^{(1)}, \kappa(m)^{(1)}) \end{aligned} \quad (3)$$

$\kappa(m)^{(1)}$  is not firable in  $a$ , therefore (3)  $< \deg^+(\kappa(m)^{(1)})$ . It follows easily from (2) and (3) that (1) holds.

Let  $f : U \rightarrow \mathbb{N}$  be given by

$$f(y) = \begin{cases} E(\kappa(m_1)^{(1)}, \kappa(m_2)^{(1)}) & \text{if } y = e_{m_1, m_2} \text{ for some} \\ & m_1, m_2 \in M \text{ and } \kappa(m_1)^{(1)} \neq \kappa(m_2)^{(1)} \\ N & \text{if } y = e_{m_1, m_2} \text{ for some} \\ & m_1, m_2 \in M \text{ and } \kappa(m_1)^{(1)} = \kappa(m_2)^{(1)} \\ \min\{ \sum_{x \in M \setminus M_a} f(e_{x, m}) : a \in \mathfrak{U}_m \} & \text{if } y = w_m \text{ for some } m \in M \text{ and} \\ & \text{and } \mathfrak{U}_m \neq \{\mathbf{0}\} \\ 1 & \text{if } y = w_m \text{ for some } m \in M \text{ and} \\ & \text{and } \mathfrak{U}_m = \{\mathbf{0}\} \end{cases}$$

, where  $U$  is the collection of all variables of  $\Omega$ . The proof is completed by showing that  $f$  is a non-negative integer solution of  $\Omega$ . Since  $CFG(G, \mathcal{O})$  is a ASM, it follows easily that for any two distinct elements  $m_1, m_2 \in M$ , if  $e_{m_1, m_2}$  and  $e_{m_2, m_1}$  both are in  $U$  then  $f(e_{m_1, m_2}) = f(e_{m_2, m_1})$ . It remains to show that for each  $m \in M$ ,  $f$  satisfies  $\mathfrak{C}(m)$ . If  $\mathfrak{U}_m = \{\mathbf{0}\}$  then the assertion follows easily. If  $\mathfrak{U}_m \neq \{\mathbf{0}\}$ , it is straightforward to verify that  $f(w_m) \leq \sum_{x \in M \setminus M_a} f(e_{x, m})$  for any  $a \in \mathfrak{U}_m$ . We are left with proving  $\sum_{x \in M \setminus M_a} f(e_{x, m}) < f(w_m)$  for any

$a \in \mathfrak{U}_m$ . For this purpose, we show that  $\sum_{x \in M \setminus M_a} f(e_{x,m}) < \sum_{x \in M \setminus M_b} f(e_{x,m})$  for any  $b \in \mathfrak{U}_m$ . We have

$$\begin{aligned}
& \sum_{x \in M \setminus M_a} f(e_{x,m}) = \\
&= \sum_{\substack{x \in M \setminus M_a \\ \kappa(x)^{(1)} \neq \kappa(m)^{(1)}}} f(e_{x,m}) + \sum_{\substack{x \in M \setminus M_a \\ \kappa(x)^{(1)} = \kappa(m)^{(1)}}} f(e_{x,m}) = \\
&= \sum_{\substack{x \in M \setminus M_a \\ \kappa(x)^{(1)} \neq \kappa(m)^{(1)}}} E(\kappa(x)^{(1)}, \kappa(m)^{(1)}) + Q \times N \tag{4}
\end{aligned}$$

where  $Q = |\{(v, n) \in \kappa(M \setminus M_a) : v = \kappa(m)^{(1)}\}|$ . There are two possibilities

a.  $Q = \kappa(m)^{(2)} - 1$ . It follows from (1) and  $|\{(v, n) \in \kappa(M \setminus M_b) : v = \kappa(m)^{(1)}\}| = \kappa(m)^{(2)} - 1$  that

$$\begin{aligned}
(4) &< \sum_{\substack{x \in M \setminus M_b \\ \kappa(x)^{(1)} \neq \kappa(m)^{(1)}}} E(\kappa(x)^{(1)}, \kappa(m)^{(1)}) + (\kappa(m)^{(2)} - 1) \times N = \\
&= \sum_{x \in M \setminus M_b} f(e_{x,m})
\end{aligned}$$

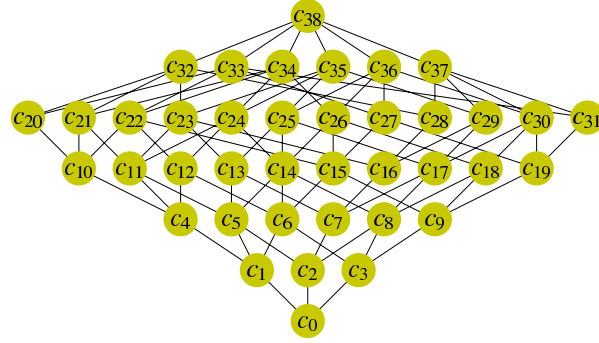
b.  $Q < \kappa(m)^{(2)} - 1$ . It follows from the definition of  $N$  that

$$\begin{aligned}
(4) &< N + Q \times N \leq (\kappa(m)^{(2)} - 1) \times N \leq \\
&\leq \sum_{\substack{x \in M \setminus M_b \\ \kappa(x)^{(1)} \neq \kappa(m)^{(1)}}} E(\kappa(x)^{(1)}, \kappa(m)^{(1)}) + (\kappa(m)^{(2)} - 1) \times N = \\
&= \sum_{x \in M \setminus M_b} f(e_{x,m})
\end{aligned}$$

Therefore,  $f$  is a non-negative integer solution of  $\Omega$  □

It's easy to see that the number of bits, which are input to the algorithms for linear programming to find a non-negative integer solution of  $\Omega$ , is  $O\left(\sum_{m \in M} |\mathfrak{U}_m| \times |X|\right) = O(|J| \times |X|)$ . By using the Karmarkar's algorithm, we can build an algorithm for the question 'is  $L$  in  $L(ASM)$ ?' in time  $O(|M|^{3.5} \times |J|^2 \times |X|^2 \times \log(|X|) \times \log(\log(|X|)))$ .

**Example 3.** Let  $L$  be the following lattice



This lattice, which is in  $L(CFG)$  but not in  $L(ASM)$ , was presented in [15] as an example of showing the class of lattices induced by ASM is strictly included in the class of lattices induced by CFG. We again present it here as an application of Theorem 4. The system  $\Omega$  of linear inequalities is  $\{w_{c_{32}} \geq 1, w_{c_{33}} \geq 1, w_{c_{37}} \geq 1, w_{c_{34}} > e_{c_{32},c_{34}} + e_{c_{36},c_{34}} + e_{c_{37},c_{34}}, w_{c_{34}} \leq e_{c_{33},c_{34}}, w_{c_{34}} \leq e_{c_{32},c_{34}} + e_{c_{35},c_{34}}, w_{c_{34}} \leq e_{c_{35},c_{34}} + e_{c_{36},c_{34}} + e_{c_{37},c_{34}}, w_{c_{35}} > e_{c_{33},c_{35}} + e_{c_{34},c_{35}} + e_{c_{37},c_{35}}, w_{c_{35}} \leq e_{c_{32},c_{35}}, w_{c_{35}} \leq e_{c_{33},c_{35}} + e_{c_{34},c_{35}} + e_{c_{36},c_{35}}, w_{c_{35}} \leq e_{c_{36},c_{35}} + e_{c_{37},c_{35}}, w_{c_{36}} > e_{c_{32},c_{36}} + e_{c_{33},c_{36}} + e_{c_{35},c_{36}}, w_{c_{36}} \leq e_{c_{33},c_{36}} + e_{c_{34},c_{36}}, w_{c_{36}} \leq e_{c_{32},c_{36}} + e_{c_{34},c_{36}} + e_{c_{35},c_{36}}, w_{c_{36}} \leq e_{c_{37},c_{36}}, e_{c_{34},c_{35}} = e_{c_{35},c_{34}}, e_{c_{34},c_{36}} = e_{c_{36},c_{34}}, e_{c_{35},c_{36}} = e_{c_{36},c_{35}}, e_{c_{34},c_{36}} = e_{c_{36},c_{34}}\}$   
 Using the algorithms for linear programming in the way as in the previous section, we know that the system has no non-negative integer solution. Therefore the lattice is not in  $L(ASM)$

**Example 4.** The game with the initial configuration presented in Figure 8a generates the lattice presented in Figure 8b. It is an example which is smaller than one presented in [15].

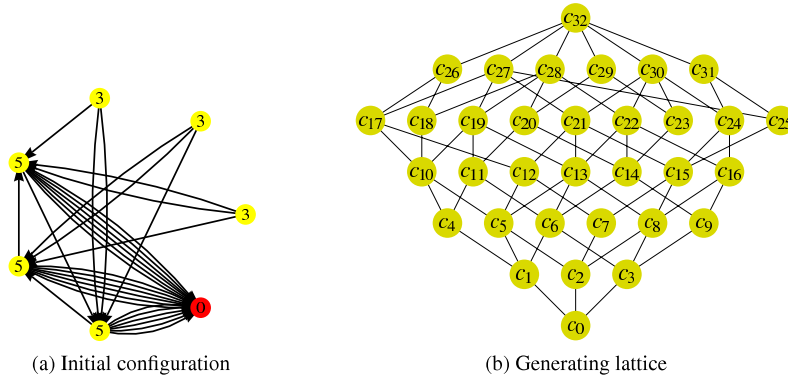


Figure 8: Smaller example

Note that two lattices in Example 3 and Example 4 are generated only by simple CFGs. It's convenient to give a sufficient condition for such lattices. The following proposition shows such a condition

**Proposition 3.** *Let  $H$  be the undirected simple graph whose vertices are  $M$  and edges are defined by  $(m_1, m_2) \in E(H)$  if  $m_1 \neq m_2$  and there are  $x_1, x_2, x_3 \in X$  such that  $x_1 < x_2, x_1 < x_3, m(x_1, x_2) = m_1$  and  $m(x_1, x_3) = m_2$ . If  $L \in L(CFG)$  and  $H$  is a complete graph then  $L$  is generated only by simple CFGs.*

The proposition is easy and left to the reader. Two above lattices are in the scope of this proposition.

In [8], the authors proved that a general CFG is always equivalent to a simple CFG. An arising question is that whether a ASM is equivalent to a simple ASM. The idea from the proof in [8] does not seem to be applicable to this model, whereas the answer follows easily from the proofs of Lemma 7 and Theorem 4

**Proposition 4.** *Any ASM is equivalent to a simple ASM.*

## 5. CFGs on acyclic graphs

In this section, we study CFGs on acyclic graphs (directed acyclic graphs). In [15], the author gave a strong relation between ASM and the simple CFGs on acyclic graphs. The author pointed out that a simple CFG on an acyclic graph is equivalent to a ASM. Here, we point out that each CFG on an acyclic graph is equivalent to a simple CFG on an acyclic graph. As a corollary, every lattice generated by a CFG on an acyclic graph is in  $L(ASM)$ . We also give a full criterion for lattices generated by CFGs on acyclic graphs.

Let  $\mathcal{G}$  be the simple directed graph whose vertices are  $M$  and edges are defined by  $(m_1, m_2) \in E(\mathcal{G})$  if and only if  $m_1 \in \bigcup_{a \in \mathfrak{U}_{m_2}} (M \setminus M_a)$ .

**Lemma 8.** *If  $L$  is generated by a CFG on an acyclic graph then  $\mathcal{G}$  is acyclic.*

*Proof.* Let  $CFG(G, \mathcal{O})$  be a CFG which generates  $L$ , where  $G$  is an acyclic directed graph. We identify the elements of  $L$  with the configurations of  $CFG(G, \mathcal{O})$ . Let  $\kappa : M \rightarrow V(G) \times \mathbb{N}$  be the map which is defined in Lemma 6. For each  $v \in V(G)$ ,  $Anc(v)$  denotes the collection of vertices  $v'$  of  $G$  such that there is a directed path from  $v'$  to  $v$  in  $G$  ( $v'$  could be equal to  $v$ ).

A sequence  $(v_1, v_2, \dots, v_k) \in V(G)^k$  is called a *valid firing sequence* if there is an execution  $\mathcal{O} = c_0 \xrightarrow{v_1} c_1 \xrightarrow{v_2} c_2 \xrightarrow{v_3} \dots \xrightarrow{v_k} c_k$  in the game. Note that if such an execution exists then it is defined uniquely. We claim that for each  $m \in M$  and each  $a \in \mathfrak{U}_m$ , we have  $\{\kappa(x)^{(1)} : x \in M \setminus M_a\} \subseteq Anc(\kappa(m)^{(1)})$ . Indeed, let  $\mathcal{O} = \mathbf{0} = c_0 \xrightarrow{v_1} c_1 \xrightarrow{v_2} c_2 \xrightarrow{v_3} \dots \xrightarrow{v_k} c_k = a \xrightarrow{\kappa(m)^{(1)}} j_m$  be an execution in the game, where  $j_m$  is the configuration which is obtained by firing  $\kappa(m)^{(1)}$  in  $a$ . It's clear that  $(v_1, v_2, \dots, v_k, \kappa(m)^{(1)})$  is a valid firing sequence. Since firing of the vertices not in  $Anc(\kappa(m)^{(1)})$  does not affect the firability of the remaining vertices, by removing all vertices not in  $Anc(\kappa(m)^{(1)})$  of the sequence, we get the sequence  $(v_{i_1}, v_{i_2}, \dots, v_{i_{k'}}, \kappa(m)^{(1)})$  which remains a valid firing sequence. There exists an execution  $\mathcal{O} = \mathbf{0} = c_0 = d_0 \xrightarrow{v_{i_1}} d_1 \xrightarrow{v_{i_2}} d_2 \xrightarrow{v_{i_3}} \dots \xrightarrow{v_{i_{k'}}} d_{k'} = a' \xrightarrow{\kappa(m)^{(1)}} j'_m$  in the game. Since the number of occurrences of  $\kappa(m)^{(1)}$  in  $(v_{i_1}, v_{i_2}, \dots, v_{i_{k'}}, \kappa(m)^{(1)})$  is the same as one in  $(v_1, v_2, \dots, v_k, \kappa(m)^{(1)})$ , it follows that  $\kappa(m(a, j_m)) = \kappa(m(a', j'_m))$ , hence  $m(a, j_m) = m(a', j'_m) = m$ . It is clear that  $sh_{a'} \leq sh_a$ , therefore  $a' \leq a$ . From the definition of  $\mathfrak{U}_m$ , we conclude that  $a' = a$ , hence  $\{\kappa(x)^{(1)} : x \in M \setminus M_a\} = \{\kappa(m(c_i, c_{i+1}))^{(1)} : 0 \leq i \leq k-1\} = \{v_i : 1 \leq i \leq k\} \subseteq Anc(\kappa(m)^{(1)})$ .

The following claim is easy and left to the reader: For each  $m \in M$  and each  $a \in \mathfrak{U}_m$ , if  $(\kappa(m)^{(1)}, n) \in \kappa(M \setminus M_a)$  then  $n < \kappa(m)^{(2)}$ .

Since  $G$  is an acyclic graph, there exists a function  $h : V(G) \rightarrow \mathbb{N}$  such that if  $(v_1, v_2) \in E(G)$  then  $h(v_1) < h(v_2)$ . Let  $N = \max\{sh_1(v) : v \in V(G)\}$ . We define  $h' : M \rightarrow \mathbb{N}$  by  $h'(m) = N \times h(\kappa(m)^{(1)}) + \kappa(m)^{(2)}$ . To prove  $\mathcal{G}$  is acyclic, it suffices to show that for every  $(m_1, m_2) \in \mathcal{G}$ , we have  $h'(m_1) < h'(m_2)$ . From the definition of  $\mathcal{G}$ , there exists  $a \in \mathfrak{U}_{m_2}$  such that  $m_1 \in M \setminus M_a$ . There are two possibilities

- a.  $\kappa(m_1)^{(1)} = \kappa(m_2)^{(1)}$ . It follows from the second claim that  $h'(m_1) = N \times h(\kappa(m_1)^{(1)}) + \kappa(m_1)^{(2)} < N \times h(\kappa(m_2)^{(1)}) + \kappa(m_2)^{(2)} = h'(m_2)$ .

- b.  $\kappa(m_1)^{(1)} \neq \kappa(m_2)^{(1)}$ . It follows from the first claim that  $\kappa(m_1)^{(1)} \in \text{Anc}(\kappa(m_2)^{(1)})$ . Therefore  $h'(m_1) = N \times h(\kappa(m_1)^{(1)}) + \kappa(m_1)^{(2)} \leq N \times h(\kappa(m_2)^{(1)}) + \kappa(m_2)^{(2)} - N + (\kappa(m_1)^{(2)} - \kappa(m_2)^{(2)}) < N \times h(\kappa(m_2)^{(1)}) + \kappa(m_2)^{(2)} = h'(m_2)$

□

We recall an result in [15]

**Theorem 5.** [15] *Let  $C$  be a simple CFG on an acyclic graph  $G$ . Then  $C$  is equivalent to a ASM.*

Here, our main result of this section

**Theorem 6.** *Any CFG on an acyclic graph is equivalent to a ASM.*

*Proof.* Let  $CFG(G, \mathcal{O})$  be a CFG such that  $G$  is an acyclic graph.  $L$  denotes  $CFG(G, \mathcal{O})$ . By Theorem 3 that for each  $m \in M$ ,  $\mathcal{E}(m)$  has non-negative integer solutions. Let  $U_m$  be the collection of all variables of  $\mathcal{E}(m)$  and  $f'_m : U_m \rightarrow \mathbb{N}$  be a non-negative integer solution of  $\mathcal{E}(m)$ . The function  $f_m : U_m \rightarrow \mathbb{N}$  defined by

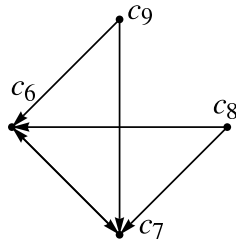
$$f_m(y) = \begin{cases} f'_m(y) & \text{if } y = w \text{ or } y \in \{e_x : x \in \bigcup_{a \in \text{In}_m} (M \setminus M_a)\} \\ 0 & \text{otherwise} \end{cases}$$

is also a non-negative integer solution of  $\mathcal{E}(m)$ . By using solutions  $f_m$ , it follows from the construction of the CFG in the proof of Theorem 3 that  $L$  is generated by a simple CFG on a graph, say  $G'$ , such that  $V(G') = M \cup \{s\}$  and if  $(v_1, v_2) \in E(G')$  then  $v_2 = s$  or  $(v_1, v_2) \in E(\mathcal{G})$ . It follows directly from Lemma 8 that  $\mathcal{G}$  is acyclic, so is  $G'$ . By Theorem 5,  $CFG(G, \mathcal{O})$  is equivalent to a ASM □

Using Lemma 8 and a similar argument as in the proof of Theorem 6, we get easily a full characterization for lattices generated by CFGs on acyclic graphs

**Corollary 3.** *Let  $L \in L(CFG)$ . Then  $L$  is generated by a CFG on an acyclic graph if and only if  $\mathcal{G}$  is acyclic.*

Let  $L(ACFG)$  denote the class of lattices generated by CFGs on acyclic graphs. Theorem 6 implies that  $L(ACFG) \subseteq L(ASM)$ . We consider the lattice shown in Figure 3a. In this case,  $\mathcal{G}$  is presented by the following figure



$\mathcal{G}$  is not acyclic, therefore the lattice is not in  $L(ACFG)$ . From Example 2, the lattice is in  $L(ASM)$ . It implies that  $L(ACFG) \subsetneq L(ASM)$ . Furthermore the lattice presented in Figure 9 is generated by a CFG on acyclic graph but not a distributive lattice. Thus  $D \not\subseteq L(ACFG)$ .

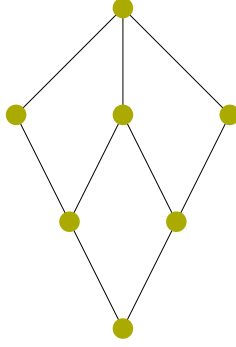


Figure 9: A lattice in  $L(ACFG) \setminus D$

## 6. Conclusion and perspectives

In the paper, we have studied three classes of lattices generated by CFGs and pointed out the relation between these classes. We gave a full characterization for each class of lattices. All characterizations we presented here are practical since it can be used to build an algorithm for the determination of the studying classes of lattices. Moreover, we get a finer chain of the classes of lattices

$$D \subsetneq L(ACFG) \subsetneq L(ASM) \subsetneq L(CFG) = L(MCFG) \subsetneq ULD$$

where  $L(MCFG)$  is the class of lattices generated by MCFGs (Multating Chip Firing Game [2, 12, 15]).

It is interesting to investigate CFGs defined on the classes of graphs which are studied widely in literature. One of these classes is the class of Eulerian directed graphs. It is a close extension of the class of undirected graphs. A graph  $G$  is *Eulerian* if it is connected and for each vertex of  $G$ , its out-degree and in-degree are equal. We define a CFG on an Eulerian graph  $G$  in the following manner. We fix a vertex  $s$  of  $G$  which will play a role as the sink of the game. We remove all out-edges of  $s$ , then we obtain the graph  $G'$  which remains a connected graph and has no closed component. The game is played on this graph. Let  $L(ECFG)$  denote the class of lattices generated by CFGs on Eulerian graphs. It is clear that  $L(ASM) \subseteq L(ECFG) \subseteq L(CFG)$ . A more precise relation between these classes of lattices remains to be done.

From the study in this paper, it turns out to be interesting that a CFG defined on each studying class of graphs is equivalent to a simple CFG which again is defined on this class. It means that in the studying classes of graphs, the simple CFGs give a full description of their classes of lattices. At the moment, we are interested in a characterization of the classes  $\mathfrak{G}$  of graphs such that simple CFGs are closed, *i.e.* for a CFG defined on  $\mathfrak{G}$ , we always can find a simple CFG which also is defined on a graph in  $\mathfrak{G}$  and equivalent to the old one. For instance, how about the class of Eulerian graphs? Are simple CFGs closed in this class?

Finally, we are interested in the following computational problem: Given a graph  $G$  and a ULD lattice  $L$ , is  $L$  generated by a CFG on  $G$ ? The characterizations presenting here do not imply that they are applicable to this problem.

So now, we have the practical characterizations for the classes of lattices generated by CFGs defined on three classes of graphs which are studied widely in literature, they are acyclic graphs, undirected graphs, and general graphs. We believe that our method presented here is not only applicable to these classes but also applicable to many other classes of graphs on which CFGs define.

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