Lattices generated by Chip Firing Game: characterizations and recognition algorithm

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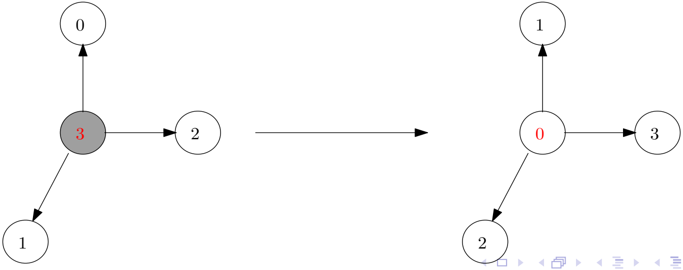
Chip firing games (CFG) on a graph: Definition

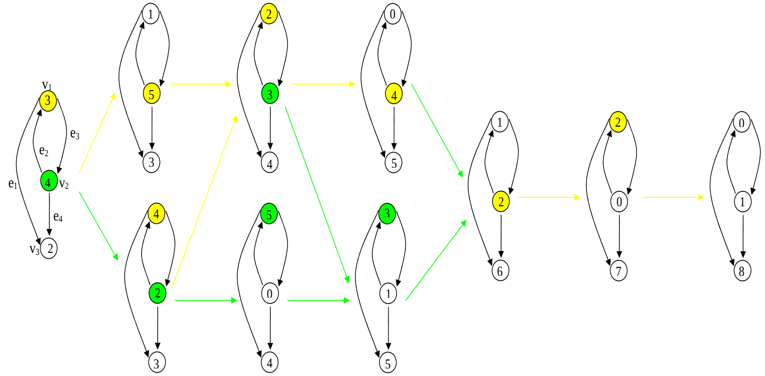
Given $G = (V, E)$ directed multigraph. A model of chip firing games is described by

(i) Configurations: each configuration is a distribution of chips on V.

(ii) Firing rule: One vertex can fire if its chips is at least its out-degree. When it fires, it gives one chip to each of its neighbors.

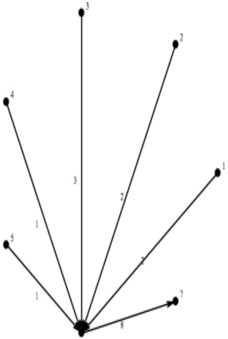


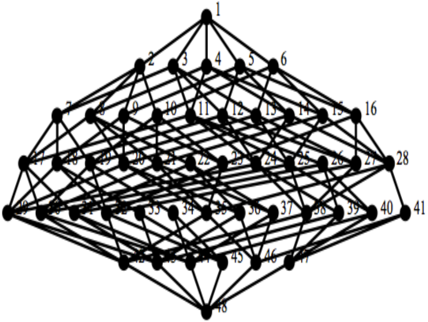
Small example of configuration space of CFG



Example of configuration space of CFG

CFG on a graph of 7 vertices





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- Chip Firing Game is a discrete dynamical model defined by D. Dhar (1990) and by A. Björner, L. Lovász and W. Shor (1991).

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- This model has applications in physics, computer science, social science and pure mathematics, etc.
- This model relates to combinatorial and algebraic objects such as lattice, matroid, spanning tree, group, etc.

Chip firing games as model to study dynamical systems

- Vectorial space: Spencer (1983)
- Automata: Durand-Lose (1999) Masson and Formenti (2006)
- Economy: Dollar game: N. Biggs (1999);
- Discrete dynamical systems: Self-Organized Criticality: Bak, Tang and Weisenfeld(1987), Latapy, Morvan and P. (1998, 2004), Goles and Kiwi (1993), ...

Chip firing games as a tool to study structures of graphs, lattices, matroids

- Algebra: Sandpile group: Dhar et al. (1995), Biggs (1999), Cori, Le Borgne and Rossin (2000)
- Structures on graphs, Matroid: C. Merino (2001); S. Oh (2010).
- CFG and Rotor-router: Levine, Propp et al. (2007, 2008)
- CFG and random spanning trees: Baker and Shokrieh (2011)
- CFG and ULD lattices: Knauer and Felsner (2009 - 2011)

Some Questions

- Under which condition, the game stops after a finite steps ?
- Does the game converge or not? (From an initial configuration, the game reaches a unique stable configuration?)
- In the case of convergent game, what is the behavior of all possible evolutions ?
- Is there any characterization or algorithm to detect the structure of such a space configuration ? (Yes: Lattice structure !)

Some Questions (2)

- In the case of convergent game, how one can know which is the stable configuration of a given initial configuration.
- Reachabality problem: given two configurations, how one can know if one is reachable from another.
- From one initial configuration, the system converges to a stable configurations. How one can say about the set of all stabe configurations ?
- Critical configurations: special stable configurations (which can be recurrent under some condition).
- What is the structure of the set of all critical configuration.
- How one can deduce properties of a graphs from its critical configurations.

Two directions to study CFG

Vertically

- Fix an initial configuration, consider the set of all reachable configurations (the structure of configuration space).
- Study the order of this configuration space: theory of order, lattice, distributive lattice, etc.
- Find a characterization of class of lattices generated by CFG.
- Recognition algorithm of the class of lattices generated by CFG and related models.

Two directions to study CFG (in this course)

Horizontally

- Consider the set of all critical configurations.
- Group structure of the set of critical configurations: the Sandpile group.
- Relation between critical configurations and clasical objects on graphs: spanning trees, Graphic Matroid ...
- Enumeration of critical configurations: Tutte polynomial.

Convergent properties

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  For a undirected connected graph with a sink, from an initial configuration, the game converges to a stable configuration.

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- **Theorem.** (Holroyd et al., '08)
  For a directed connected graph with a global sink, from an initial configuration, the game converges to a stable configuration.

Order on configuration space of CFG

- Order on CFG: configuration b is greater than configuration a if b can be obtained from a by applying a sequence of firings. Note: This is a well defined order due from the convergent properties of CFG.

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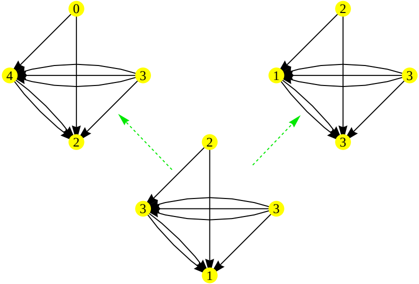
Examples of configuration space of CFG



A step of evolution

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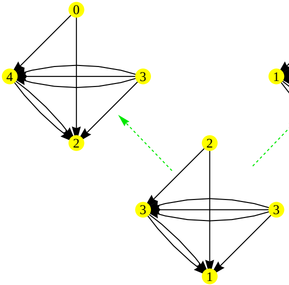

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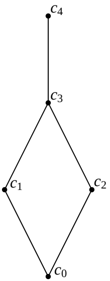
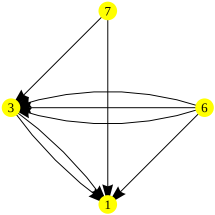


Generating lattice

Examples of configuration space of CFG (2)



Changing initial configuration

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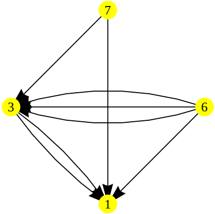

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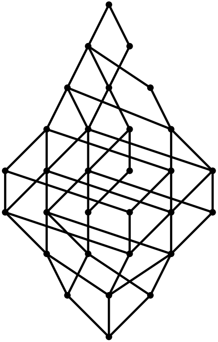

Generating lattice

Order, Lattice, Distributive lattice, ULD

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- In a partial order set, for two elements a and b, the infimum of a and b, $inf(a, b)$ (if it exists) is the maximum element which is smaller than a and b. Duality for supremum of a and b: $sup(a, b)$.

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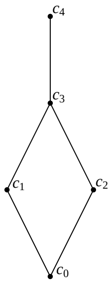
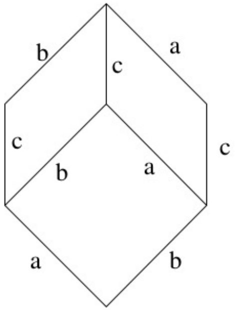
- A distributive (D) lattice is a lattice such that two operators inf and sup have distributive property.
- A upper locally distributive (ULD) lattice is lattice such that: for any element a and if b is the sup of all upper covers of a, then the interval $[a, b]$ is a hypercube.

Example of Distributive and ULD lattice





Distributive lattices $\subsetneq$ ULD laticces

- **Distributive lattices**
  - Domino and lozenge tilings of a plane region Rémila ('04)
  - Planar spanning trees Gilmer and Litherland ('86)
  - Eulerian orientations of a planar graph Felsner ('04)
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- **Upper locally distributive lattices**
  - Subtrees of a tree Boulaye ('67)
  - Convex subsets of a poset Birkhoff and Bennett ('85)
  - Transitively oriented subgraphs of a transitively oriented digraph Björner ('85)
  - Feasible multi-sets of an antimatroid with repitition Björner and Ziegler ('92)

Lattice structure of Chip Firing Game

- Theorem (Björner and Lovász '92, Latapy and P. '01)
  For a given graph G and an initial configuration O, the configuration space $CFG(G, O)$ has a ULD lattice structure.

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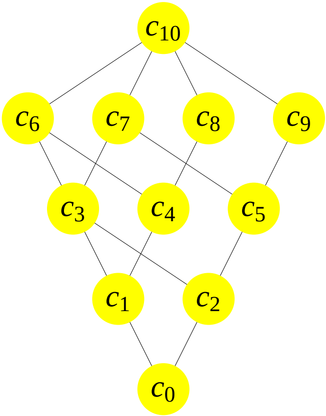
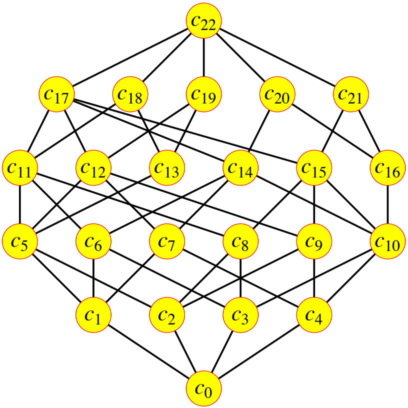
  $D \subsetneq \mathcal{L}(CFG) \subsetneq ULD$

Example of a lattice of CFG which is not in D



Example of a ULD lattice which is not a lattices of CFG



The smallest ULD lattice which is not a lattices of CFG

CFG lattice vs ULD lattice

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- **Theorem.** (Knauer '10)

  Every ULD can be represented by a simple generilized CFG.

Characterization of ULD's that can be generated by CFGs.

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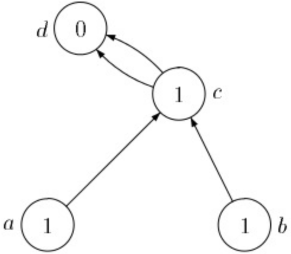
  - Find full characterizations for $\mathcal{L}(CFG)$.
  - Using CFG and related models, construct a filter from the class of distributive lattices to the class of ULD lattices.

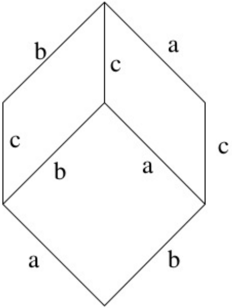
CFG vs Simple CFG

- A simple CFG is a CFG on which each vertex is fired at most once.
- Two convergent CFG are equivalent if their lattices of configuration space are isomorphic.

- **Theorem.** (Magnien, P. and Vuillon '01)
  Any convergent CFG is equivalent to a simple CFG.

Small example of CFG and ULD lattice with labeled cover relation

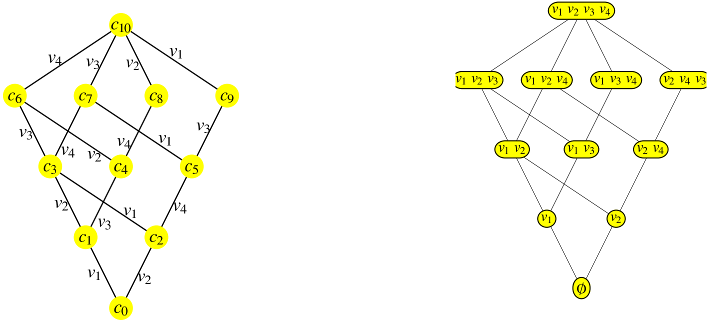




Coding each configuration of (simple) CFG by its set of firing vertices

Let L be the lattice generated by a $CFG(G, O)$:

- x of L coded by $V_x =$ firing vertices to obtained x from O.
- Cover $x \prec y$ (x obtained from y by firing v) is coded by v.



Meet irreducibles of D and ULD lattices

Definition

An element m in lattice L is called meet-irreducible if m has a unique upper cover.

Theorem. (Birkhoff '40)

A lattice is distributive if and only if it is isomorphic to the lattice of the ideals of the order induced by its meet-irreducibles.

Lemma (Caspard '98)

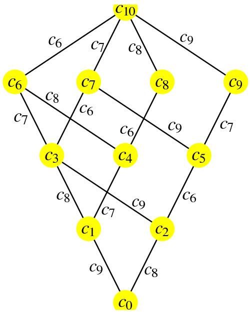
A lattice L is ULD if and only if for all $x, y \in L$,

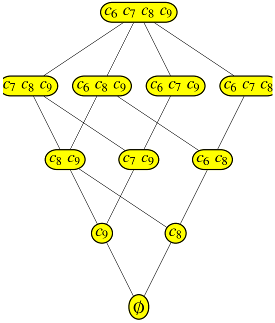
$$x \prec y \leftrightarrow M_y \subset M_x \text{ and } |M_y \backslash M_x| = 1.$$

(Here, $M_x = \{m \in M, m \geq x\}$).

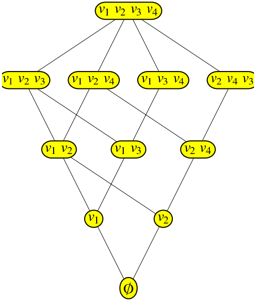
Coding each configuration of a ULD lattice a set of meet-irreducibles

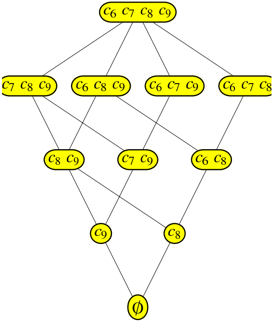
Let L be a ULD lattice, and M be its set of meet irreducibles.
- Each element x of L can be coded by $V_x = \{m \in M, m \not\geq x\}$.
- Each cover relation $x \prec y$ can be codes by $V_y \backslash V_x$.





Coding by set of firing vertices and coding by set of meet-irreducibles





Map from a CFG to a ULD lattice

Idea of the algorithm:
If CFG on G corresponds to a lattice L then
$V(G)$ corresponds to the set of meet-irreducibles of L.

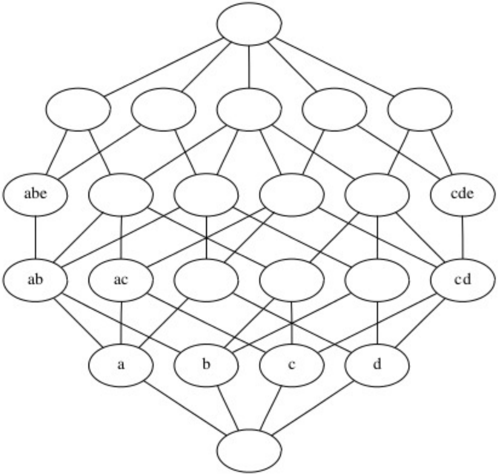
Construction of a CFG from a distributive lattice

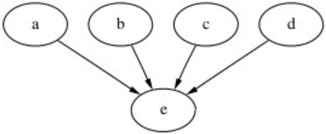
Let L be a lattice and M be its set of meet-irreducibles.

-
- The graph $G = (V, E)$ is contructed as follow:
- $V = M \cup s$
- E is convering edges in the order M, plus:
  - $indeg_M(v) - outdeg_M(v)$ edges from v to s (if it is positive)
  - one edge from v to s if v is isolated.
- The initial configuration is defined by: $c(v) = indeg_G(v) - outdeg_G(v)$

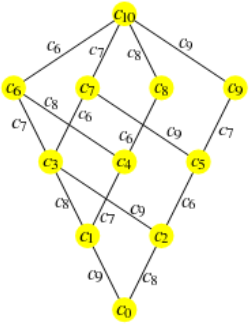
Theorem. (Magnien, P. and Vuillon '01)
$CFG(G, 0)$ is isomorphic with L.

Why a ULD can not be generated by a CFG ?



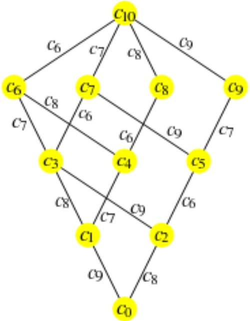


Example



In $\mathcal{E}(c_6)$, w is the number of chips needed to add to c_6 for firing it; and e_{c_8} is the number of edges from c_8 to c_6.

Example



$$\mathcal{E}(c_6) = \begin{cases} w \leq e_{c_8} \\ w \leq e_{c_7} + e_{c_9} \\ e_{c_9} < w \end{cases}$$

$$\mathcal{E}(c_8) = \mathcal{E}(c_9) = \{w \geq 1\}$$

$$\mathcal{E}(c_7) = \begin{cases} w \leq e_{c_9} \\ w \leq e_{c_6} + e_{c_8} \\ e_{c_8} < w \end{cases}$$

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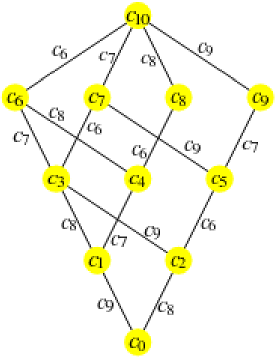

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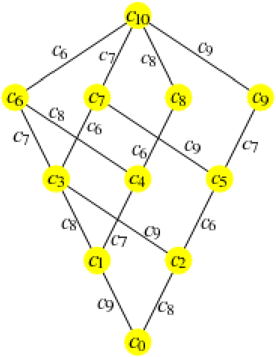

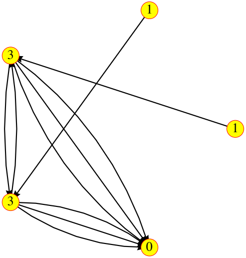

System of linear inequalities of firing process

Let $L = (X, \leq)$ be a finite lattice. For each $m \in M$

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  - $\mathfrak{U}_m = \{j^- : j \in J \text{ and } j \downarrow m\}$ (J: join irreducibles of L).
  - $\mathfrak{L}_m =$ maximals of $\{X \setminus \bigcup_{a \in \mathfrak{U}_m} \{x \in L : a \leq x\}\}$

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  - $\mathfrak{L}_m =$ maximals of $\{X \setminus \bigcup_{a \in \mathfrak{U}_m} \{x \in L : a \leq x\}\}$
- $\mathcal{E}(m) = \{ \sum_{x \in V_a} e_x < w : a \in \mathfrak{L}_m \}$

$$\cup \{w \leq \sum_{x \in V_a} e_x : a \in \mathfrak{U}_m\} \cup \{w \geq 1\}$$

(e_x is the number of edges from x to m.)

Main theorem

Theorem.
$L \in \mathcal{L}(CFG)$ if and only if for each meet- irreducible m, $\mathcal{E}(m)$ has non-negative solutions.

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$L \in \mathcal{L}(CFG)$ if and only if for each meet- irreducible m, $\mathcal{E}(m)$ has non-negative solutions.

By using Karmarkar's algorithm, we can build an algorithm to run in time

$$O(|M|^{3.5} \times |J|^2 \times |L|^2 \times log(|L|)) \times log(log(|L|))$$

Algorithm

Input : A ULD lattice L which is input as a acyclic graph with the edges defined by the cover relation
Output: **Yes** if L is in $\mathrm{L(CFG)}$, **No** otherwise. If **Yes** then give a support graph G and an initial configuration c_0 on G so that $\mathrm{CFG(G, c_0)}$ is isomorphic to L

$V(G) := M \cup \{s\}$;
$E(G) := \emptyset$;
for $m \in M$ **do**
  Construct $\mathcal{E}(m)$;
  if $\mathcal{E}(m)$ *has no non-negative integral solutions* **then Reject**;
  else
    Let f_m be a non-negative integral solution of $\mathcal{E}(m)$;
    Let U_m be the collection of all variables in $\mathcal{E}(m)$;
    for $e_x \in U_m \backslash \{w\}$ **do**
      Add $f_m(e_x)$ edges (x, m) to G
    end
    Add $f_m(w) + \sum_{e_x \in U_m \backslash \{w\}} f_m(e_x)$ edges (m, s) to G
  end
end
Construct the initial configuration c_0 by

$$c_0(v) := \begin{cases} deg^+(v) & \text{if } deg^-(v) = 0 \\ deg^+(v) - f_v(w) & \text{if } deg^-(v) \neq 0 \text{ and } v \neq s \\ 0 & \text{if } v = s \end{cases}$$

Abelian Sandpile Model (ASM) and main result

- ASM is the CFG model which is restricted to undirected graphs

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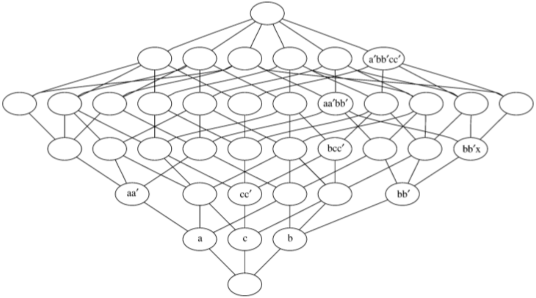
Theorem.

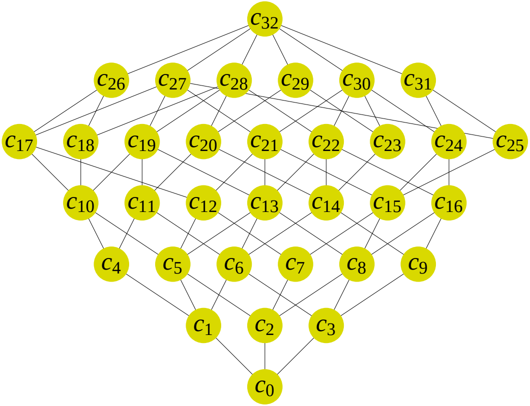
$L \in \mathcal{L}(ASM)$ if and only if $\{e_{m_1,m_2} = e_{m_2,m_1} | m_1, m_2 \in M\} \cup \bigcup_{m \in M} \mathcal{E}'(m)$ has non-negative solutions

First example of a lattice in $\mathcal{L}(CFG)$ but not in $\mathcal{L}(ASM)$



[Magnien '03]

Smallest example of a lattice in $\mathcal{L}(CFG)$ but not in $\mathcal{L}(ASM)$



CFGs on DAGs and main theorem

Theorem.

Any CFG on an acyclic directed graph is equivalent to a ASM.
Therefore $\mathcal{L}(ACFG) \subsetneq \mathcal{L}(ASM)$.

Inclusion on classes of lattices

$$D \subsetneq \mathcal{L}(ACFG) \subsetneq \mathcal{L}(ASM) \subsetneq \mathcal{L}(CFG) \subsetneq ULD$$

Next problems

- What about $\mathcal{L}(ECFG)$? (CFG on Eulerian graphs)
- A characterization of classes of graphs which are closed by simple CFGs
- Given a pair (G, L), is L generated by a CFG on G for some initial configuration?

Main references

K. Meszaros Y. Peres J. Propp A. E. Holroyd, L. Levin and D. B. Wilson.
Chip firing and rotor-routing on directed graphs.
In and Out of Equilibrium II. Progress in Propability (Birkhauser 2008), 60.

A. Bjorner, L. Lovász, and W. Shor.
Chip-firing games on graphes.
E.J. Combinatorics, 12:283–291, 1991.

Phan Thi Ha Duong and Tran Thi Thu Huong.
On the stability of sand piles model.
Theoret. Comput. Sci., 411(3):594–601, 2010.

E. Goles and M.A. Kiwi.
Games on line graphes and sand piles.
Theoret. Comput. Sci., 115:321–349, 1993.

M. Latapy and H.D. Phan.
The lattice structure of chip firing games.
Physica D, 115:69–82, 2001.

C. Magnien, H. D. Phan, and L. Vuillon.
Characterisation of lattice induced by (extended) chip firing game.
Discrete Math. Theoret. Comput. Sci., AA:229–244, 2001.

Thi Ha Duong Phan.
Two sided sand piles model and unimodal sequences.
ITA, 42(3):631–646, 2008.

Trung Van Pham and Thi Ha Duong Phan.
Lattices generated by chip firing game models: characterizations and recognition algorithm

Thank you for your attention!