Mealy Automata, Singular Points and Wang Tilings

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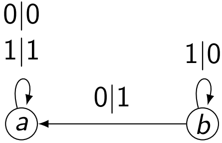
Journées SDA2, 3 juillet 2017

ANR JCJC 12 JS02 012 01

How to Generate Groups with a (Mealy) Automaton



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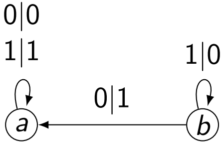

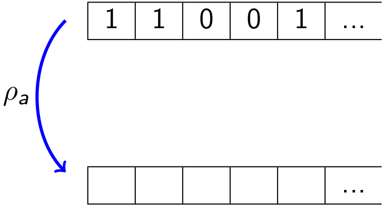

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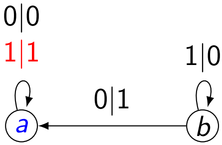

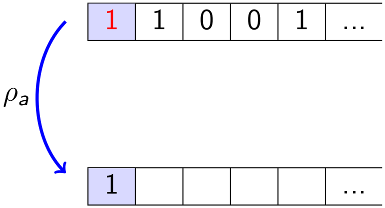

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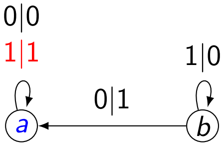

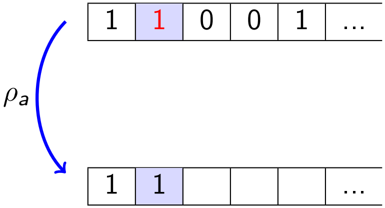

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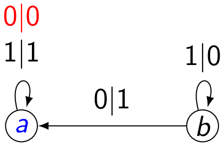

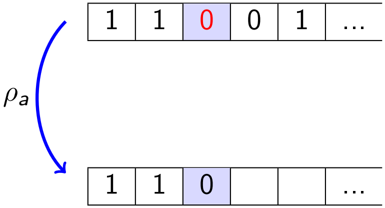

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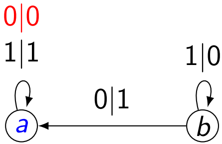

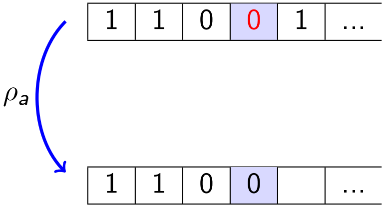

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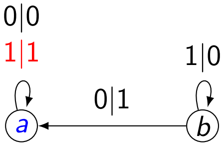

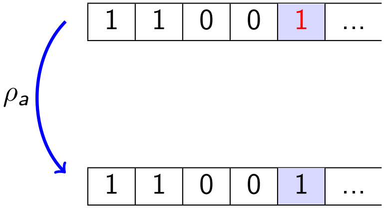

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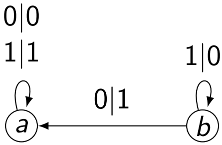

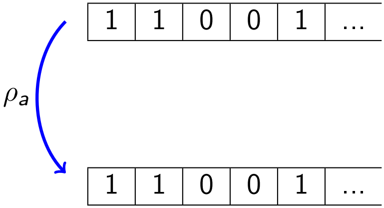

$\rho_a : \Sigma^* \to \Sigma^*$

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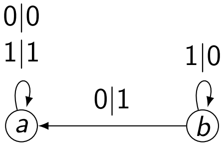

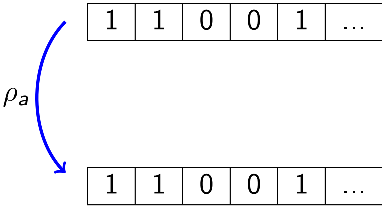

$$\rho_a : \Sigma^* \to \Sigma^*$$
$$u \mapsto u$$

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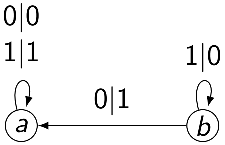

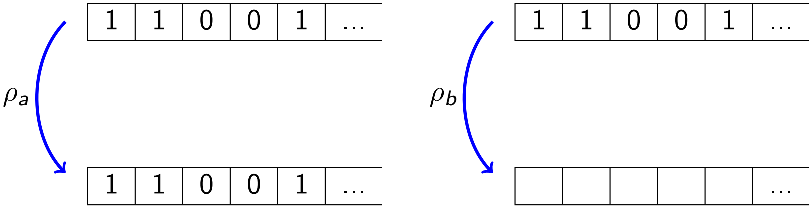

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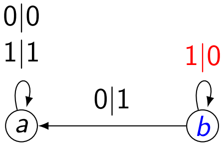

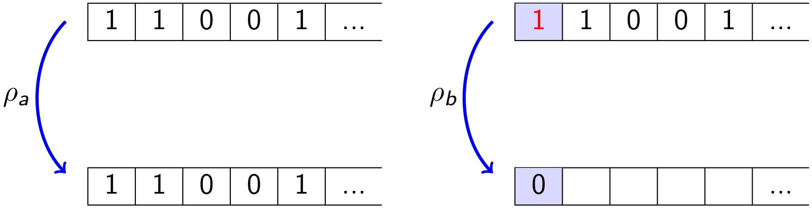

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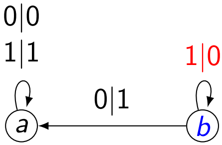

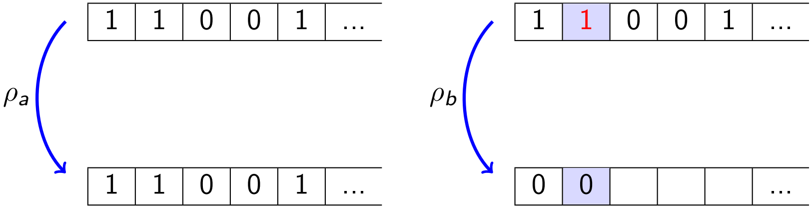

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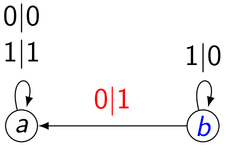

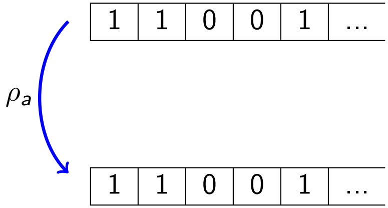

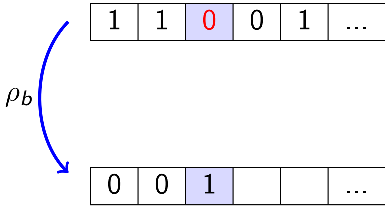

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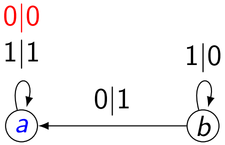

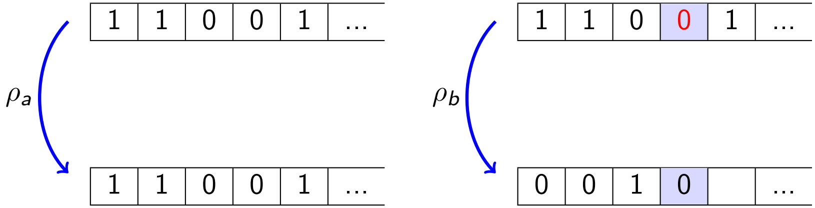

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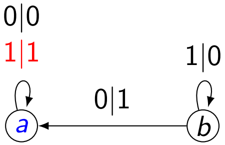

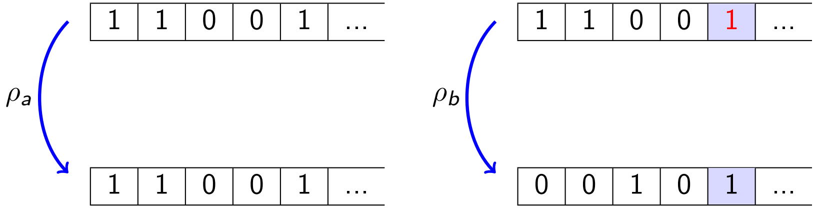

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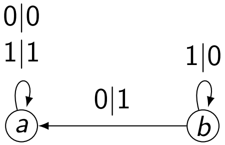

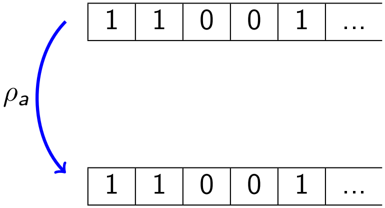

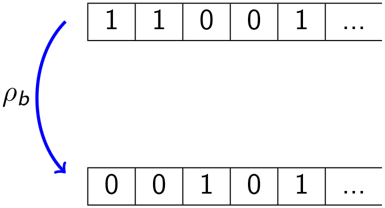

$$\rho_a : \Sigma^* \to \Sigma^*$$
$$u \mapsto u$$

$$\rho_b : \Sigma^* \to \Sigma^*$$

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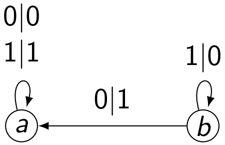
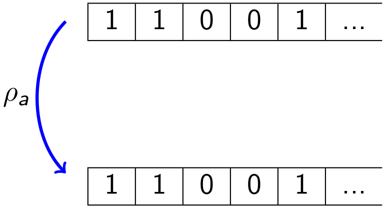
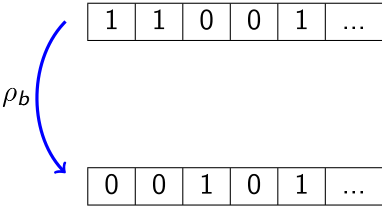




$$\rho_a : \Sigma^* \to \Sigma^*$$
$$u \mapsto u$$



$$\rho_b : \Sigma^* \to \Sigma^*$$
$$u \mapsto u + 1$$

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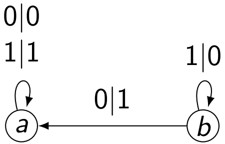
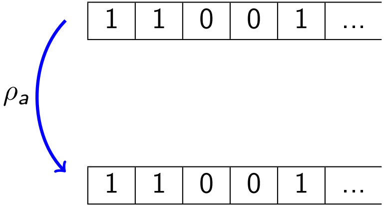
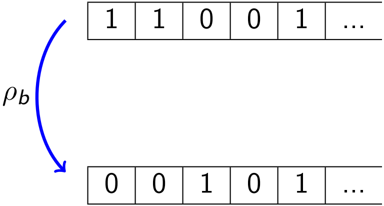


$\langle \mathcal{A} \rangle = \langle \rho_a, \rho_b \rangle$

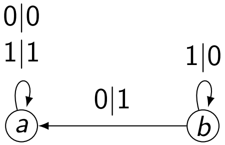




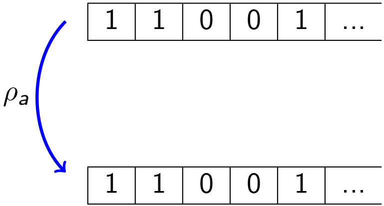
$$\rho_a : \Sigma^* \to \Sigma^*$$
$$u \mapsto u$$

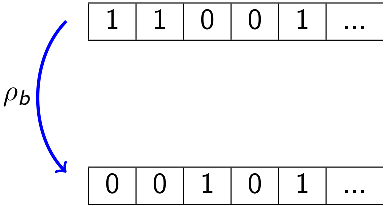
$$\rho_b : \Sigma^* \to \Sigma^*$$
$$u \mapsto u + 1$$

How to Generate Groups with a (Mealy) Automaton



$\langle \mathcal{A} \rangle = \langle \rho_a, \rho_b \rangle = \langle \rho_a, \rho_b, \rho_a^{-1}, \rho_b^{-1} \rangle$





$$\rho_a : \Sigma^* \to \Sigma^*$$
$$u \mapsto u$$

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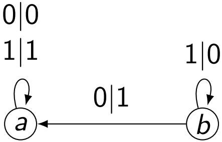
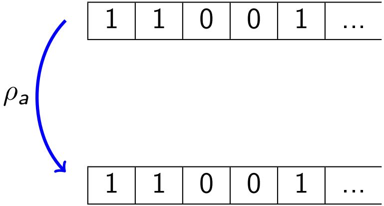
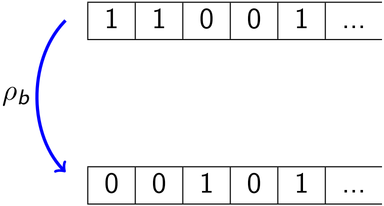


$$\langle \mathcal{A} \rangle = \langle \rho_a, \rho_b \rangle = \langle \rho_a, \rho_b, \rho_a^{-1}, \rho_b^{-1} \rangle \; \simeq \mathbb{Z}$$





$$\rho_a : \Sigma^* \to \Sigma^*$$
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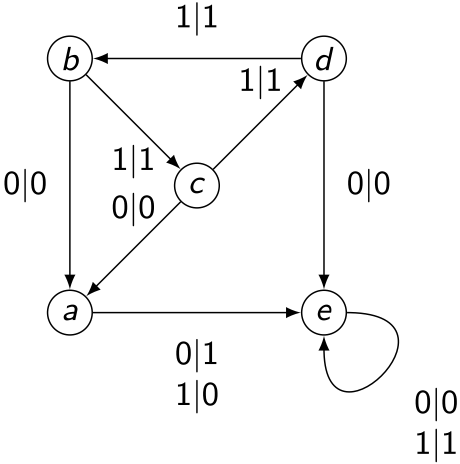
Another Mealy automaton



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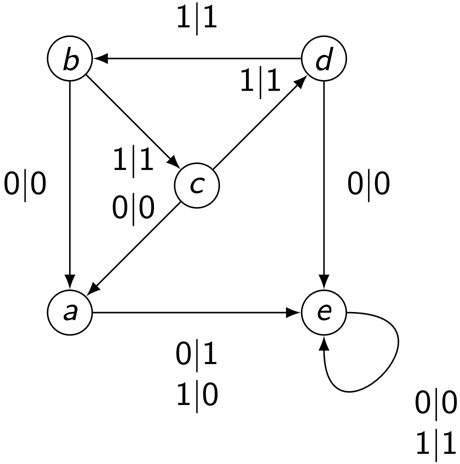
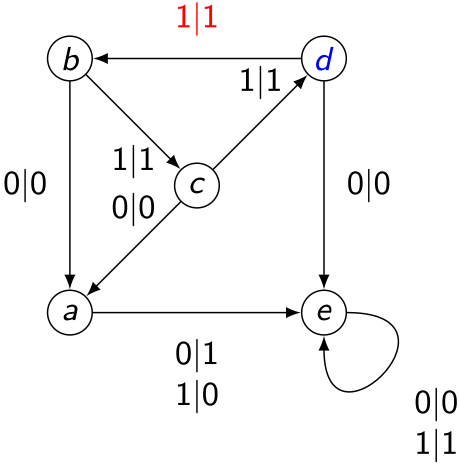


Grigorchuk automaton

Another Mealy automaton

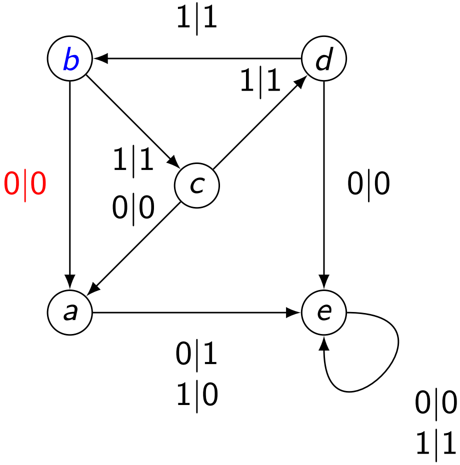


Grigorchuk automaton

$\rho_q : \Sigma^* \to \Sigma^*, \; q \in Q$

	1	0	0	0	1
d					

Another Mealy automaton



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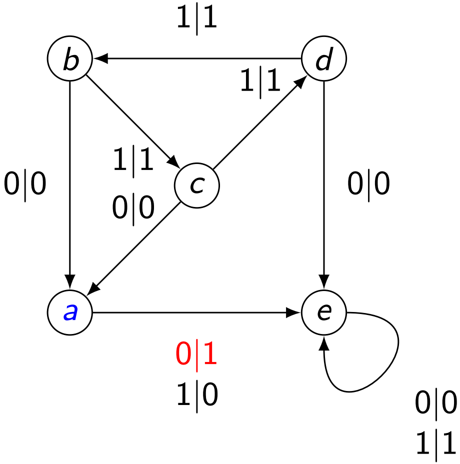

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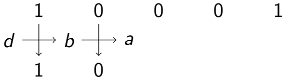

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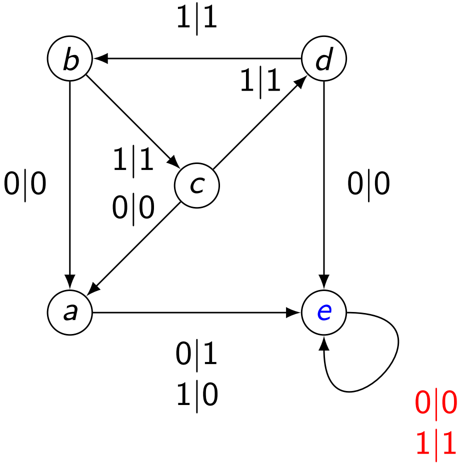

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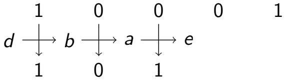

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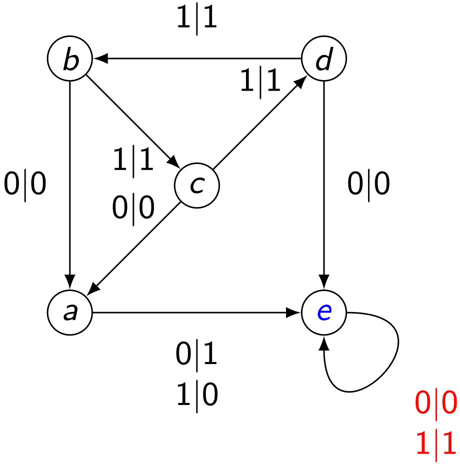

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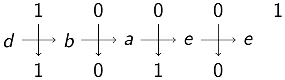

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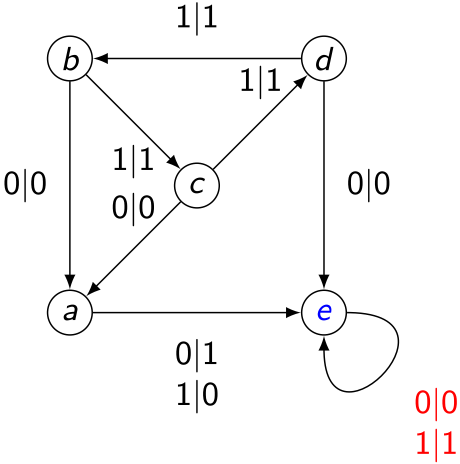

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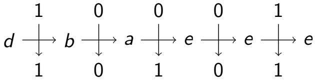

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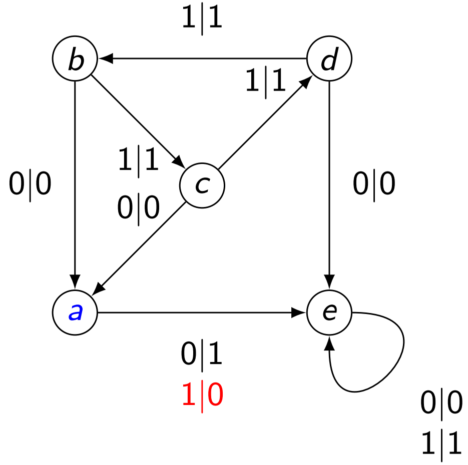

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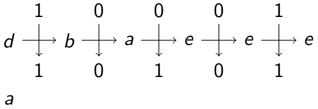

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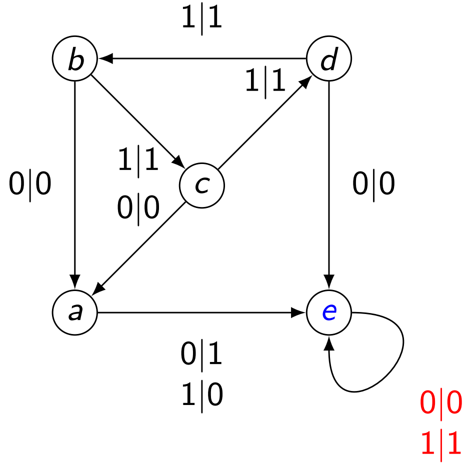

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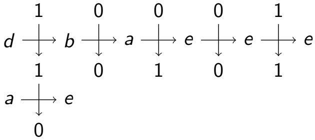

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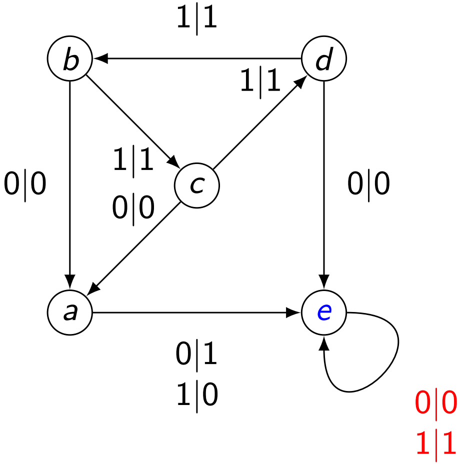

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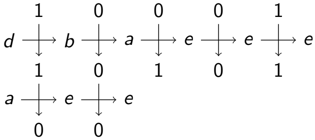

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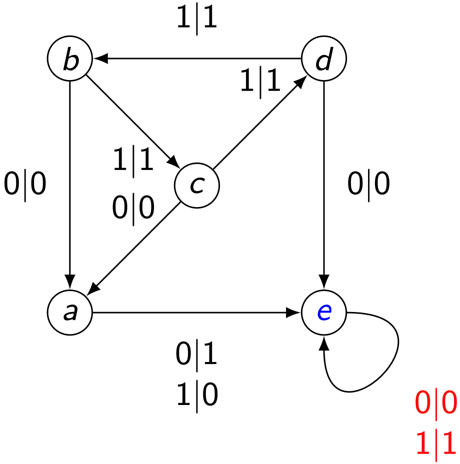

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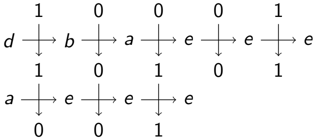

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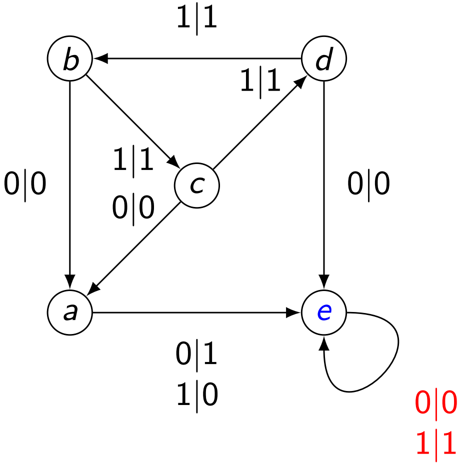
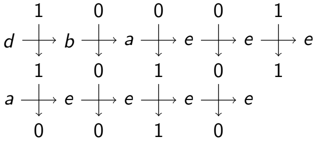


Grigorchuk automaton



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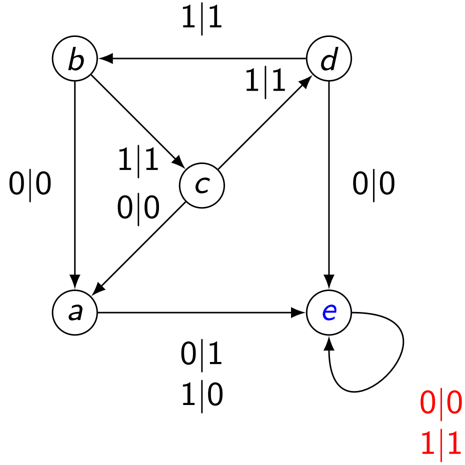

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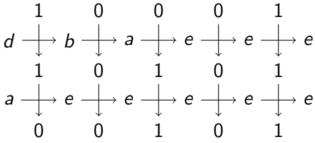

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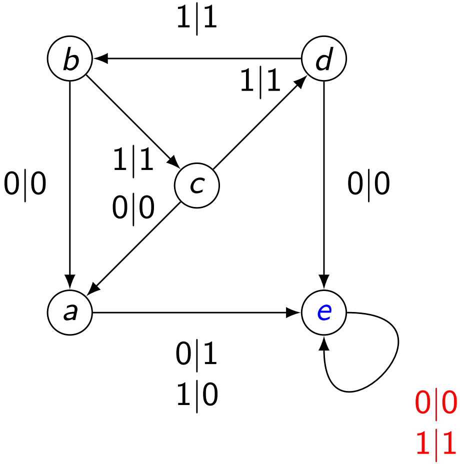

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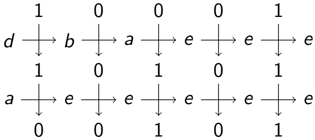

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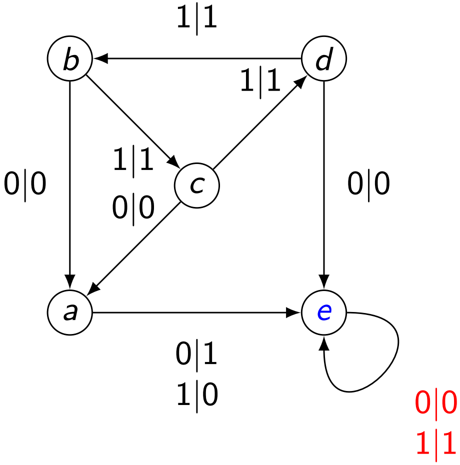

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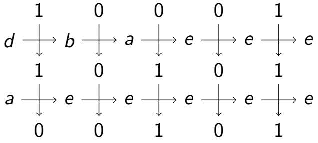

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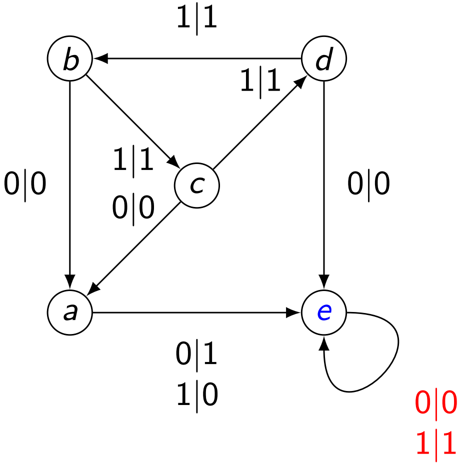

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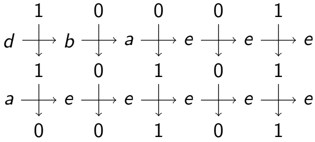

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$da \in \mathcal{A}^2$

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Order of an element

$x \in G$ has finite order if $\exists n \geq 1, x^n = \mathrm{e}$

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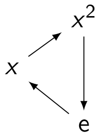

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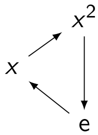
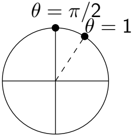


$x^3 = e$, x is torsion

- $\mathbb{Z}/n\mathbb{Z}$ is torsion (all its elements have finite order)
- $\mathbb{Z}$ is torsion-free (0 is the only element of finite order)
- On the circle $\mathbb{R}/2\pi\mathbb{Z}$; $\pi/2$ has finite order but 1 has infinite order



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Burnside Problem (1902):

Can a finitely generated group have all elements of finite order and be infinite?

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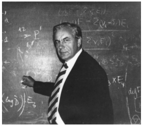

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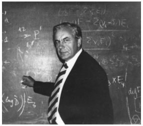

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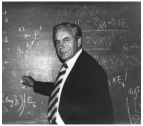

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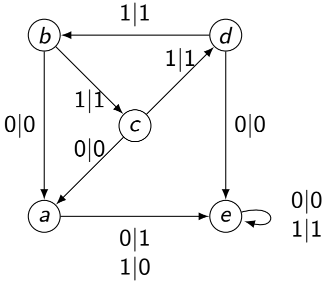

Growth

Cayley Graph: $\Gamma(G, S)$

$g \xrightarrow{s} h \qquad g.s = h,\ g, h \in G,\ s \in S$

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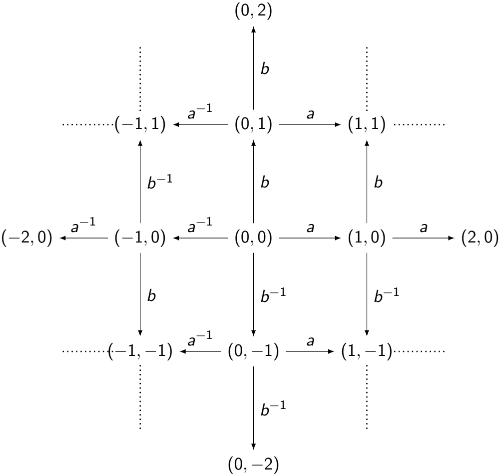

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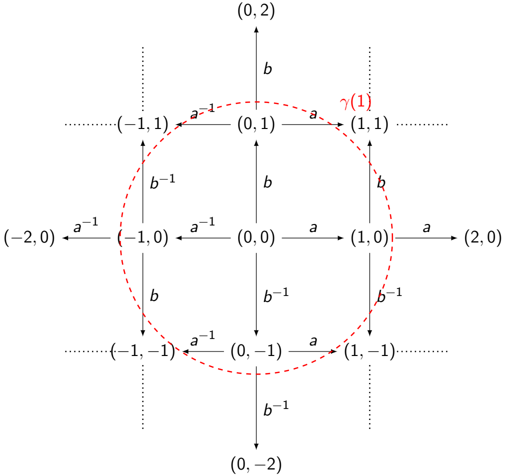

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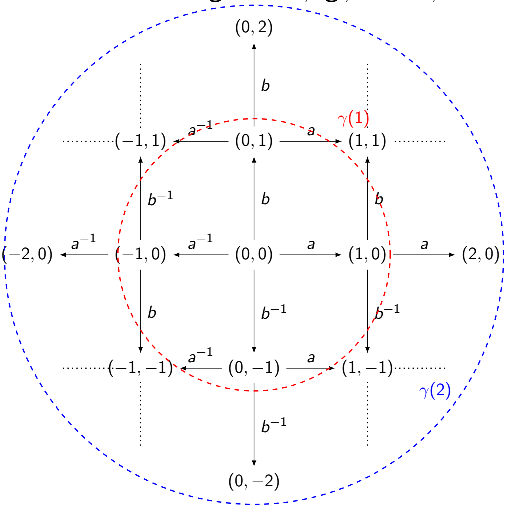

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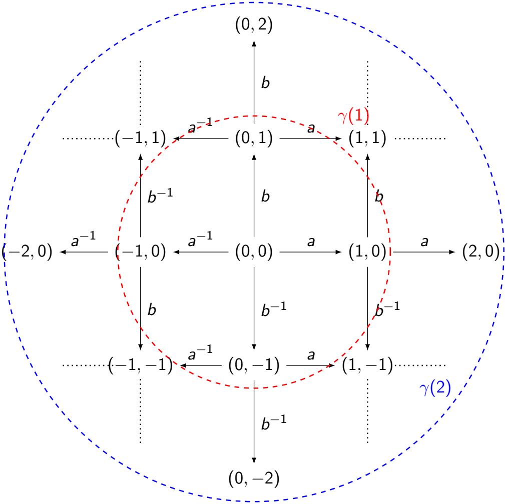

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$\vdots$

$\gamma(n) = 2n^2 + 2n + 1$

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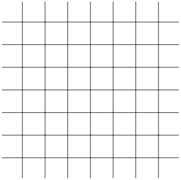

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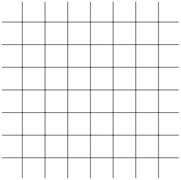

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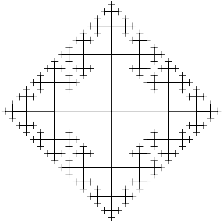

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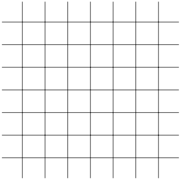

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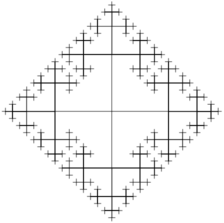

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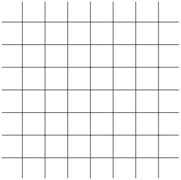

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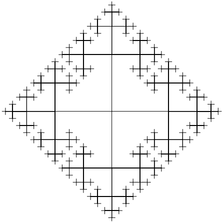

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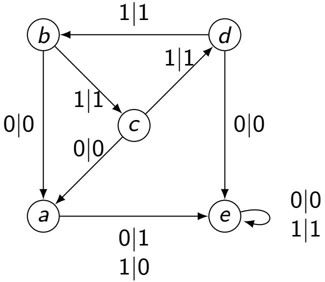

$$e^{n^{0.51}} \leq \gamma(n) \leq e^{n^{0.77}}$$

And more!

How to get a strongly aperodic SFT in the Grigorchuk group

Sebastián Barbieri [1]

1 : ENS DE LYON
École Normale Supérieure - Lyon

An action of a finitely generated group over a Cantor set is called effectively closed if there is a Turing machine which recieves as input a cylinder and a generator and computes an effective aproximation of the complement of the image of such cylinder under the generator. I will show that every effectively closed action of a finitely generated group G can be realized as a factor of the G-subaction of an SFT in $G \times H_1 \times H_2$ for any pair of infinite f.g. groups H_1,H_2. As a corollary we obtain that any group of the form $G_1 \times G_2 \times G_3$ admits a strongly aperiodic SFT whenever all the G_i are finitely generated and have decidable word problem. In particular, we show how this theorem implies the existence of strongly aperiodic SFTs in the Grigorchuk group.

Type :	:	contribution soumise
Thématiques	:	SDA*
PDF version	:	PDF version

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- finitely generated

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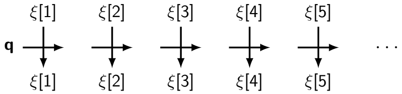

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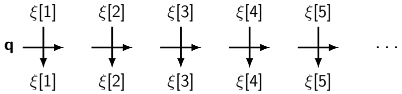
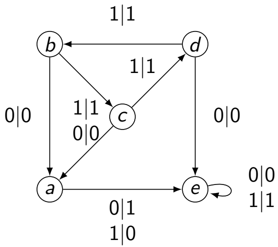


Example



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studied by Y. Vorobets

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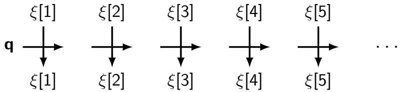

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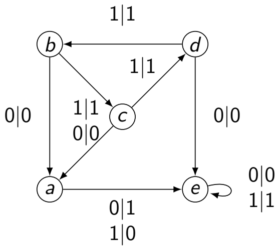

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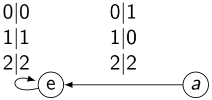
Interesting elements



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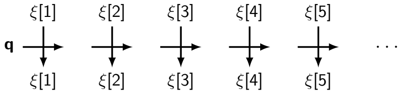

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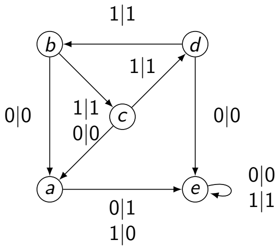

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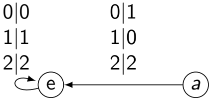
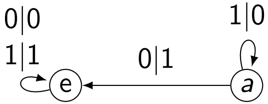


2^ω is stabilised by a
Avoid ending in e

Action on a regular rooted tree



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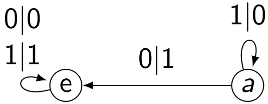

Set of words $\Sigma^* \simeq$ regular rooted tree T

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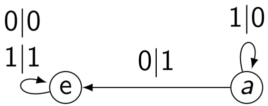

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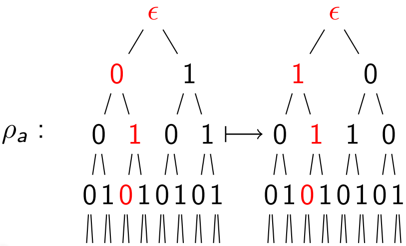

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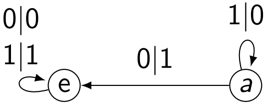

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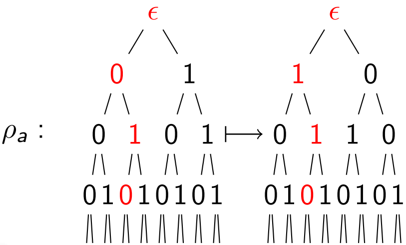

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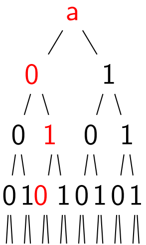
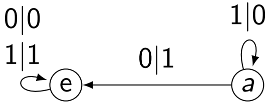
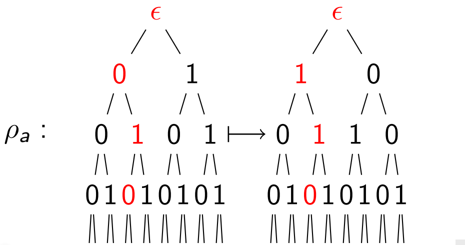


Action on a regular rooted tree



Set of words $\Sigma^* \simeq$ regular rooted tree T

Action



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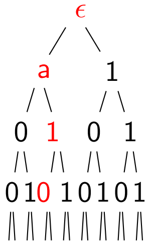

$\rho_a(0) = 1,\ \delta_0(a) = \mathrm{e}$

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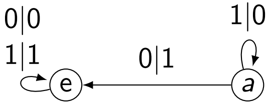

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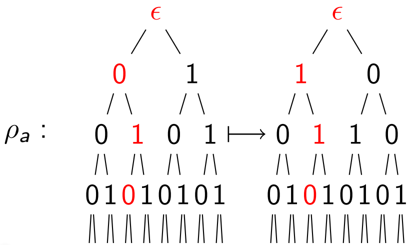

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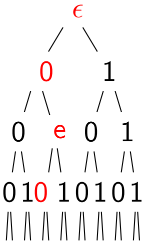

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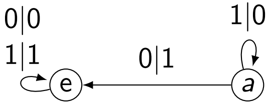

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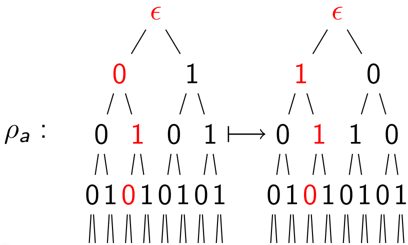

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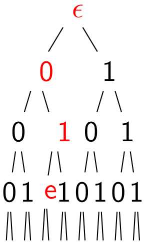

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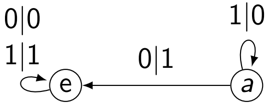

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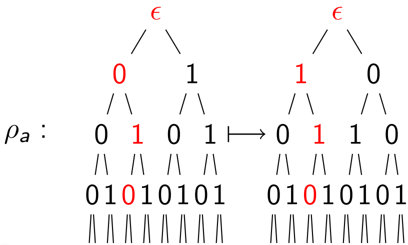

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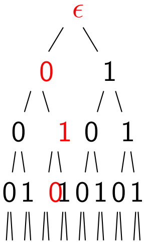

Singular points

$$St : \xi \mapsto \mathsf{Stab}_{\langle \mathcal{A} \rangle}(\xi)$$

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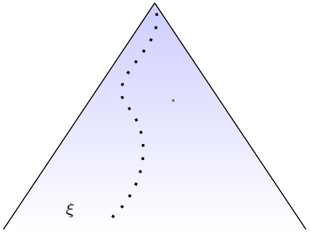

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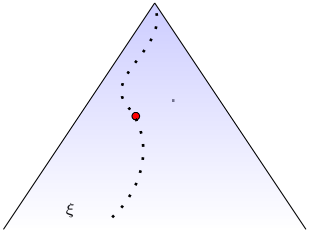

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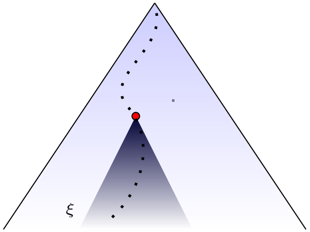

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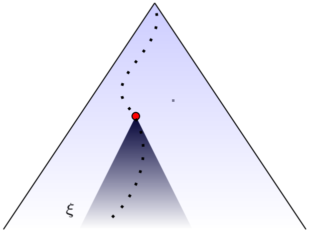

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ξ singular $\Leftrightarrow$ St is not continuous in ξ

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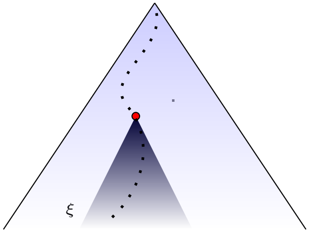

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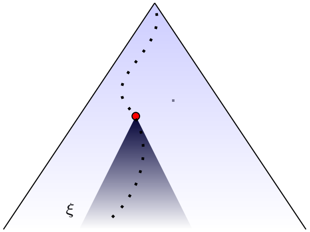

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The set of singular points has measure 0.

Characterising singular points

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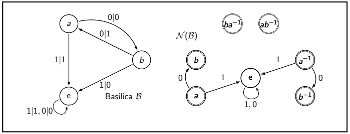
Expressing Sing as a language:

(fractal) contracting automata



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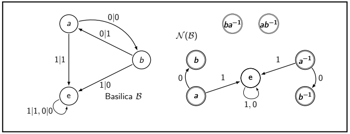
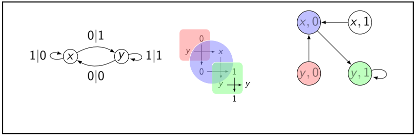


Consider specific stabilisers, via commuting pairs:

(bi)reversible automata



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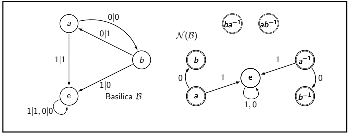

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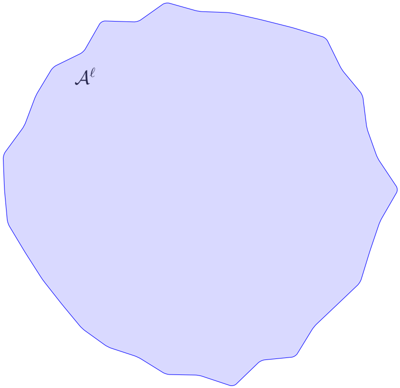

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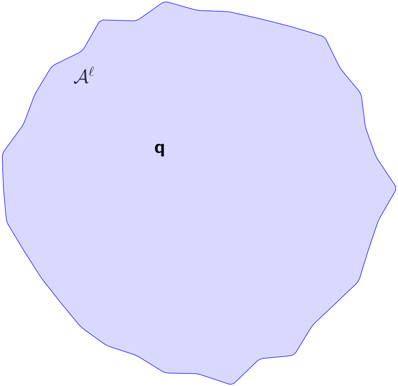

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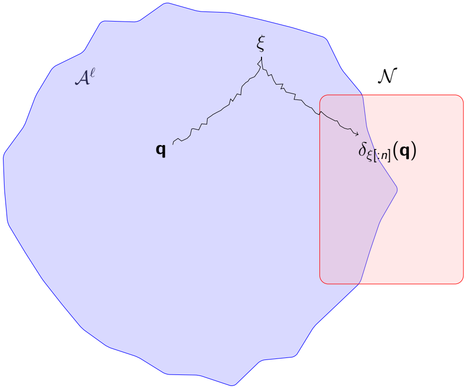

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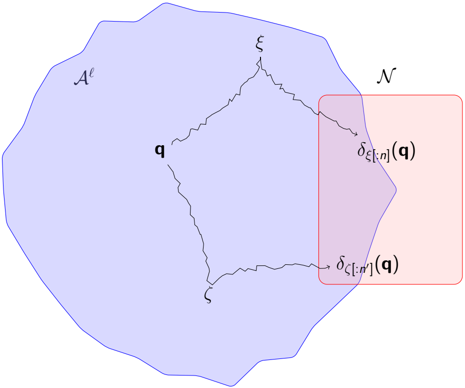

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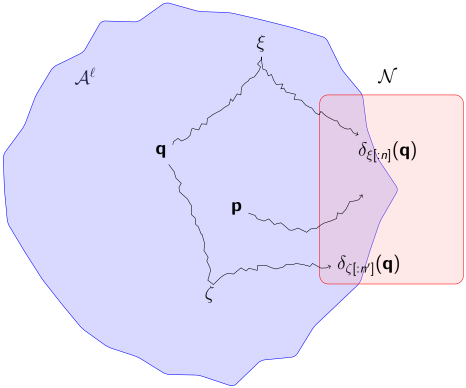

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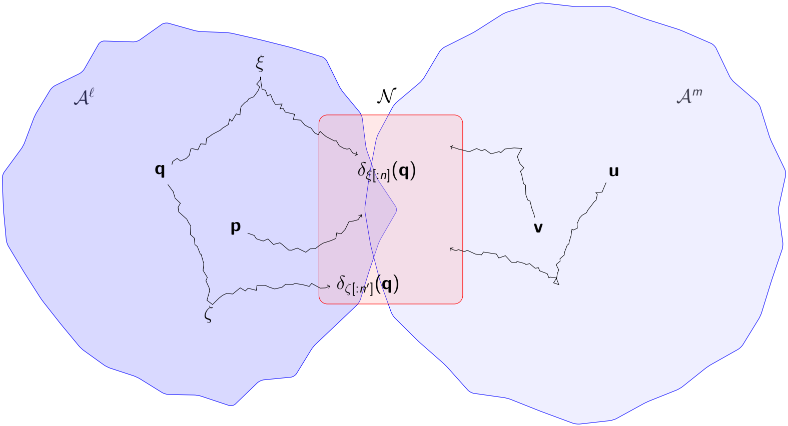

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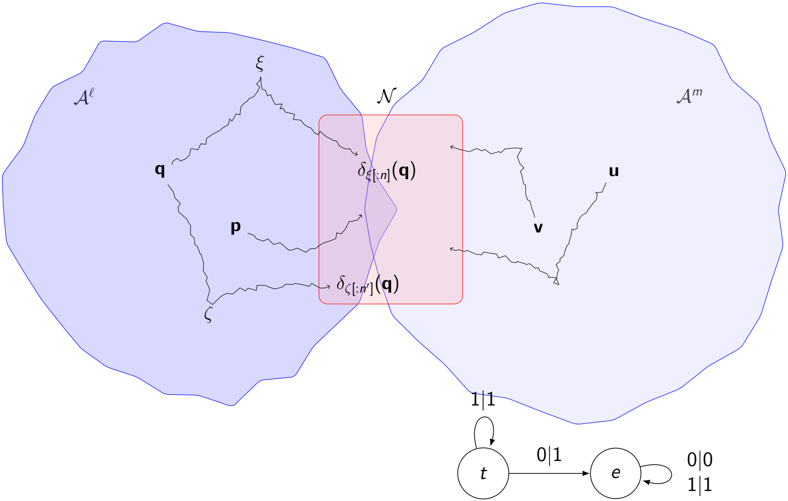

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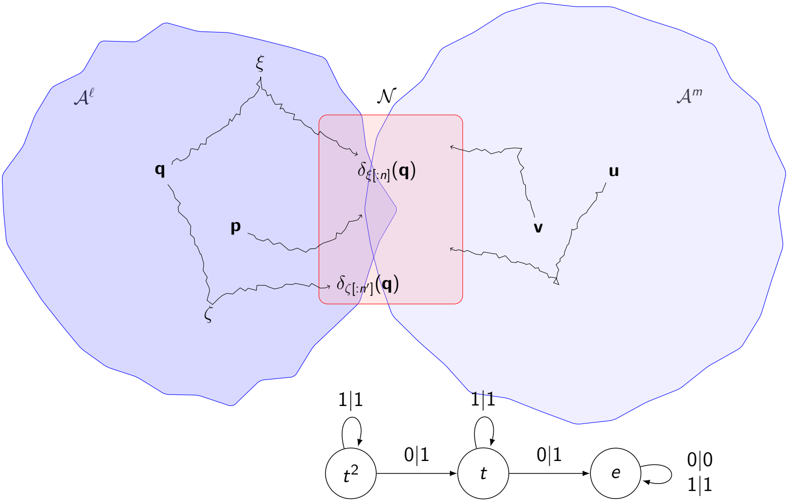

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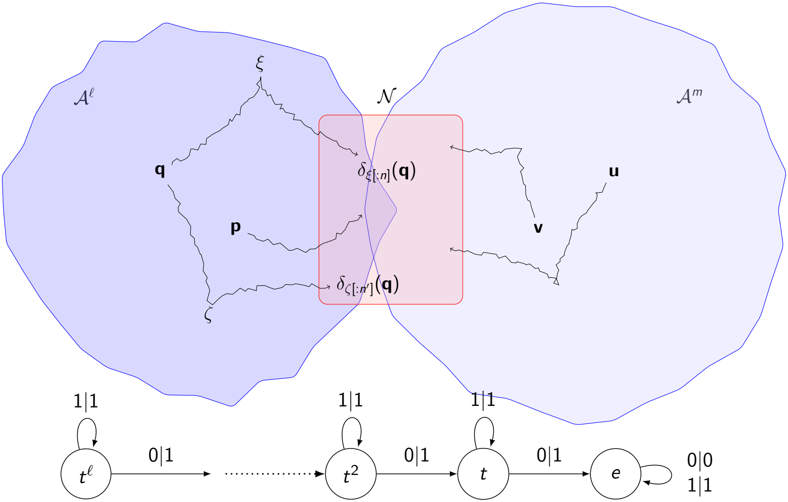
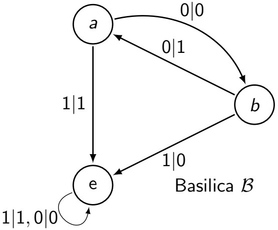
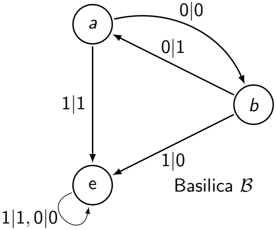


Contracting Automata and singular points



Basilica $\mathcal{B}$

Contracting Automata and singular points



Basilica $\mathcal{B}$

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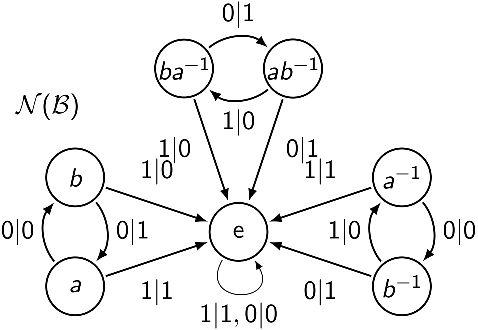

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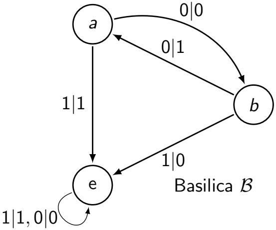

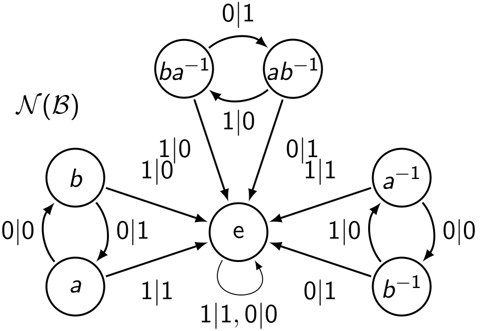

$\delta_{0^\omega}(aba)$

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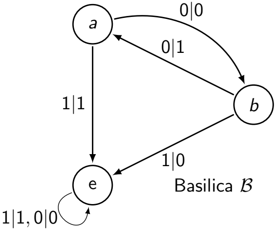

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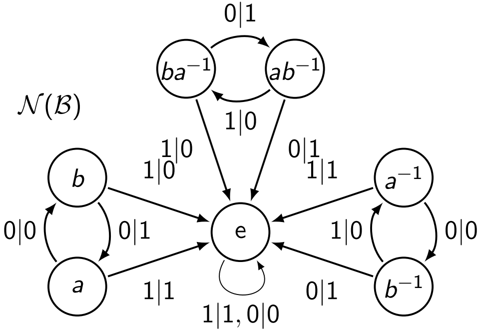

$\delta_{0^\omega}(aba) : \delta_0(aba) = ba$

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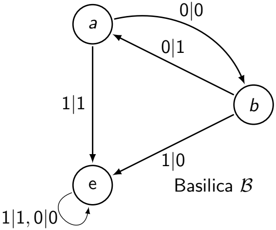

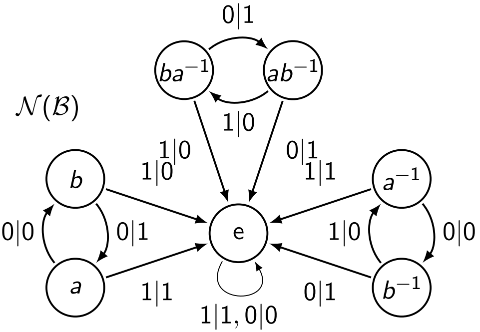

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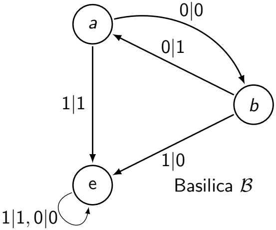

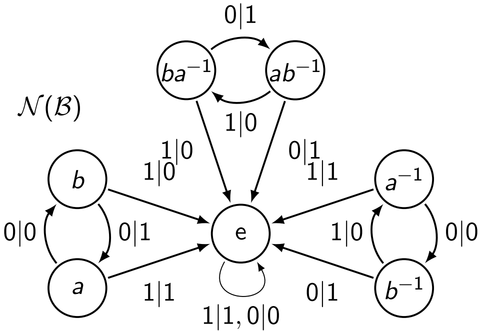

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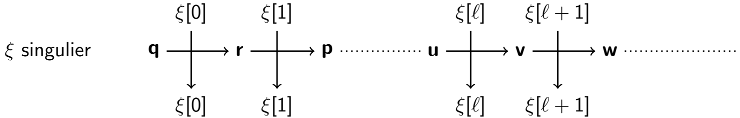

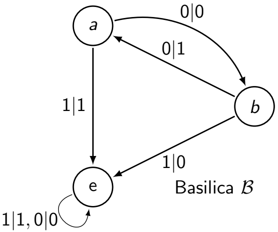

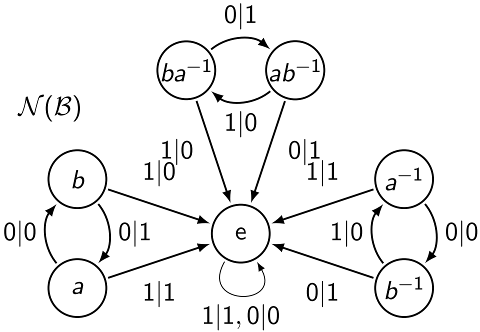

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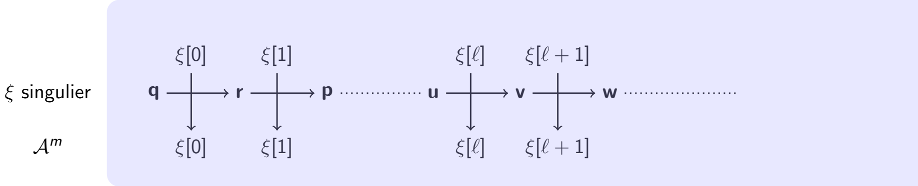

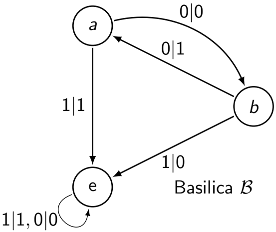

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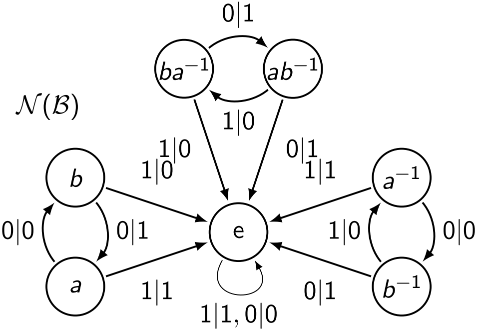

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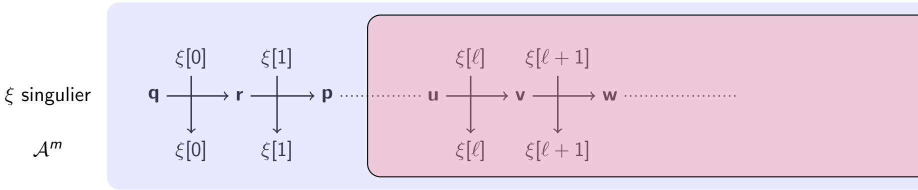

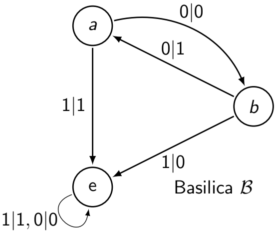

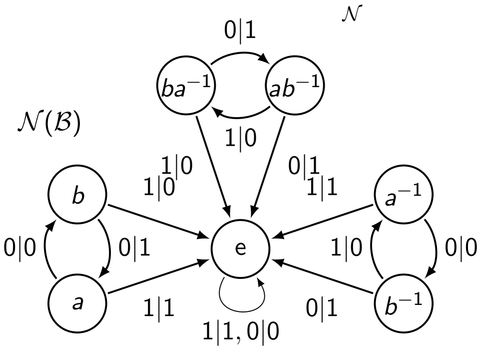

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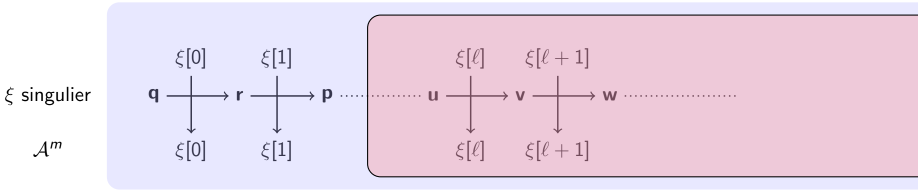

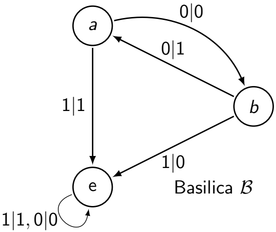

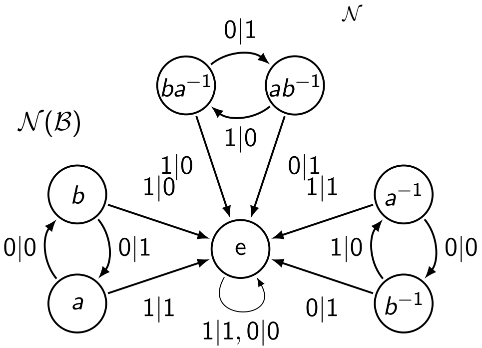

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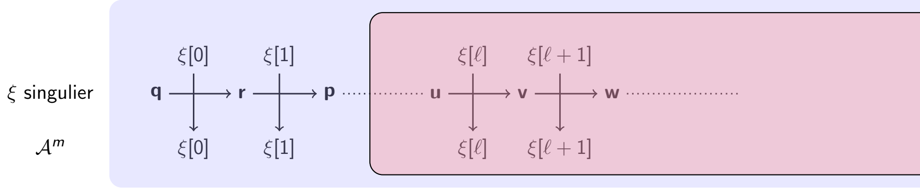

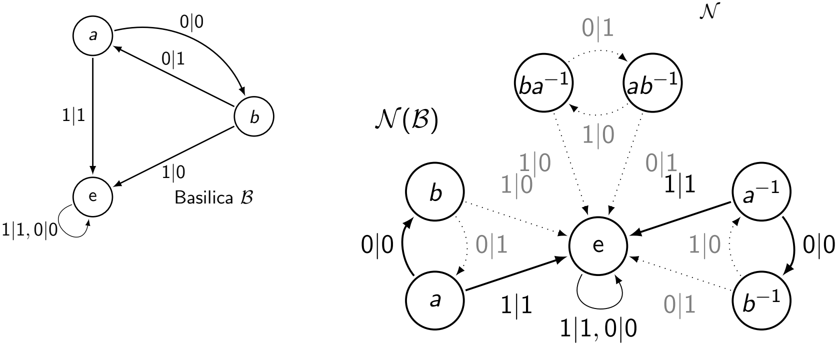

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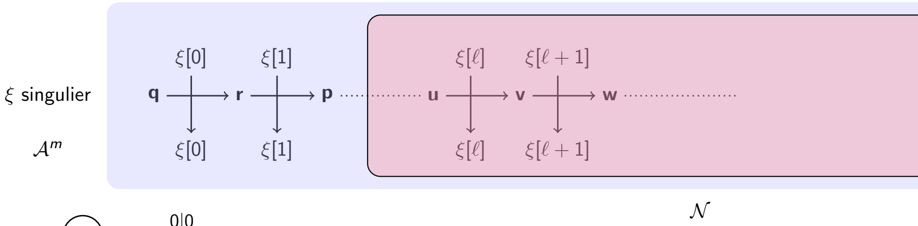
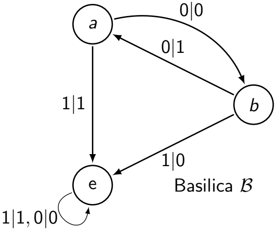
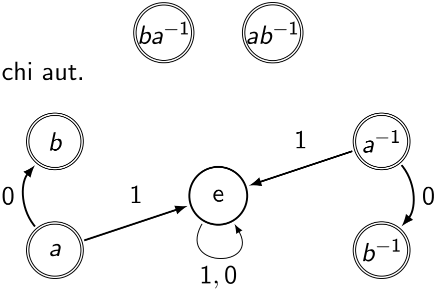




Basilica $\mathcal{B}$

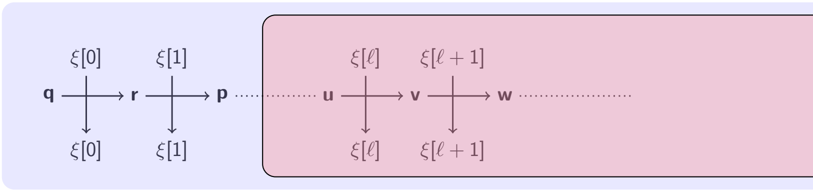
Büchi aut.

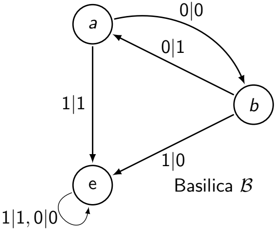


Contracting Automata and singular points

ξ singulier

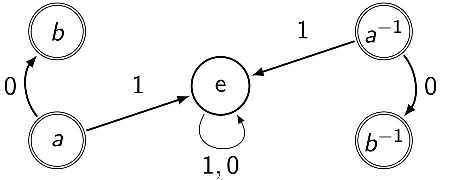
$\mathcal{A}^m$





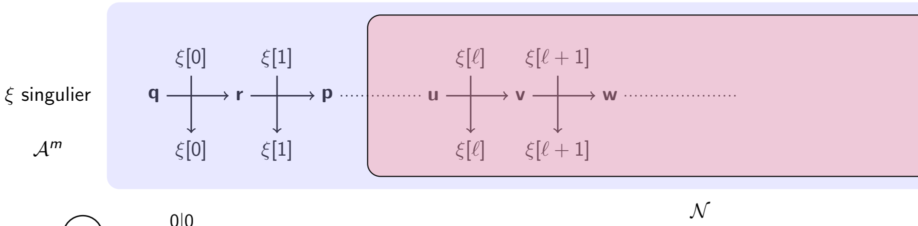
$\mathcal{N}$

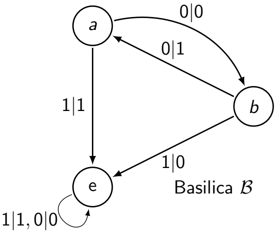
Büchi aut.



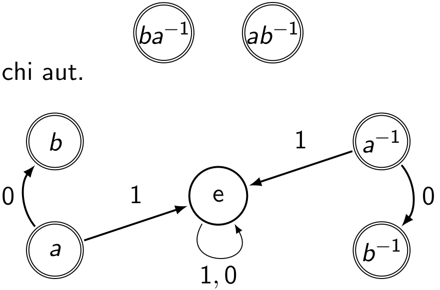
$\delta_{0^\omega}(aba) : \delta_0(aba) = ba; \delta_0(ba) = a \in \mathcal{N}(\mathcal{B})$ no path from a avoids e

Contracting Automata and singular points





Büchi aut.



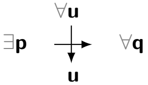
Lemma

$\mathrm{Sing}(\mathcal{B}) = \emptyset$

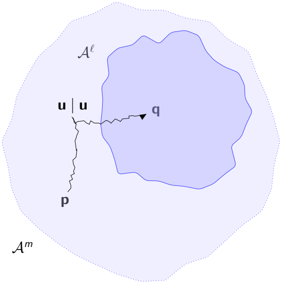
Fractal

Definition

$\mathcal{A}$ fractal $\iff \forall \mathbf{q}, \forall \mathbf{u}, \exists \mathbf{p} \in \mathrm{Stab}_{\langle \mathcal{A} \rangle}(\mathbf{u}),\ \delta_u(\mathbf{p}) = \mathbf{q}$



Fractal



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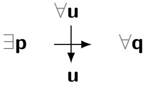
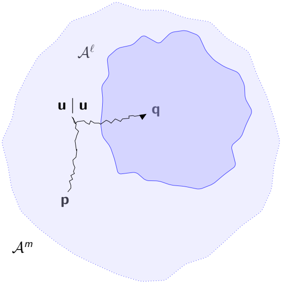


Fractal



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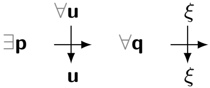
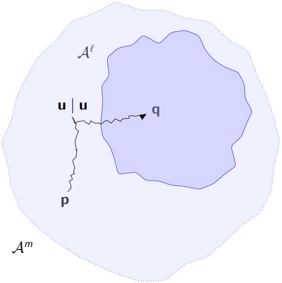
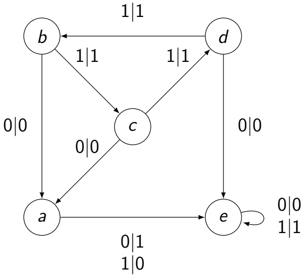


Fractal





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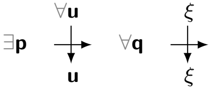
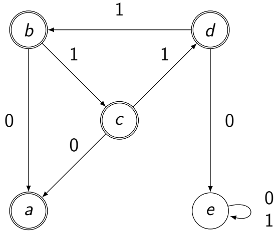
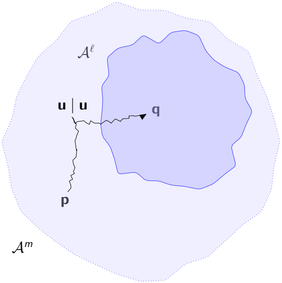
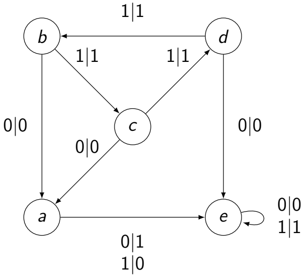




Fractal





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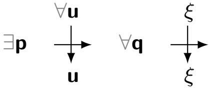
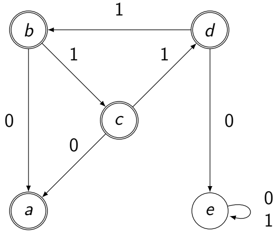
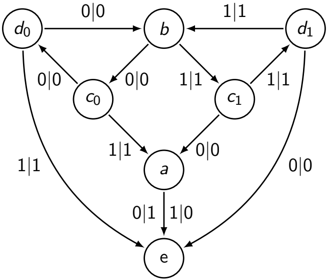




Proposition [Vorobets, AGKPR]

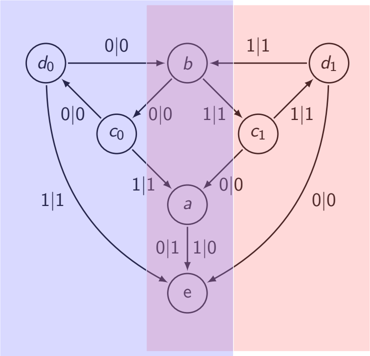
$\mathrm{Sing}(\mathcal{G}) = (0+1)^* 1^\omega$

Singular points



Automaton $\mathcal{W}$

Singular points



Automaton $\mathcal{W}$

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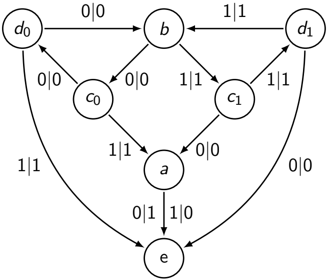


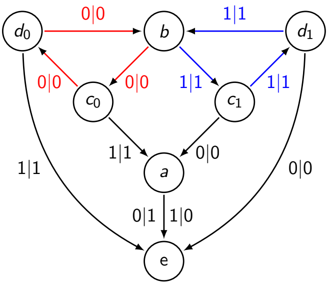
Automaton $\mathcal{W}$

- Contracting

nucleus of size 1027

- Fractal

Singular points



Automaton $\mathcal{W}$

- Contracting

nucleus of size 1027

- Fractal

Corollary

$(000 + 111)^\omega \subset \mathrm{Sing}(\mathcal{W})$

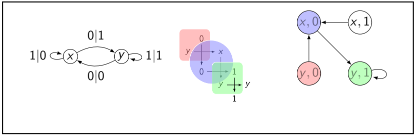
Characterising singular points

Theorem

The set of singular points has measure 0.

Consider specific stabilisers, via commuting pairs:

(bi)reversible automata



Find singular points

Lemma

$\exists\, \xi$ singular $\iff \exists \mathbf{u}^{\omega}$ singular

Find singular points

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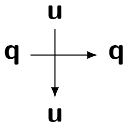
Lemma

$\exists \mathbf{u}, \mathbf{q},\ \delta_{\mathbf{u}}(\mathbf{q}) = \mathbf{q},\ \rho_{\mathbf{q}}(\mathbf{u}) = \mathbf{u}$



Find singular points

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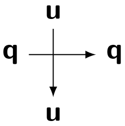


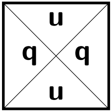
Definition

$(\mathbf{q}, \mathbf{u})$ commuting pair

Find singular points

Lemma

$\exists\, \xi$ singular $\iff \exists \mathbf{u}^\omega$ singular



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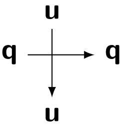

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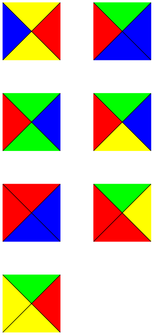
$(\mathbf{q}, \mathbf{u})$ commuting pair

Wang tilings and transducers



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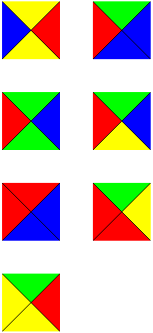

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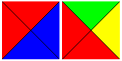

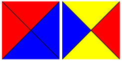

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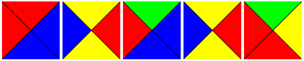

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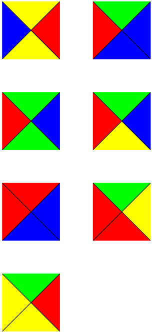

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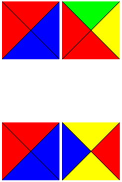

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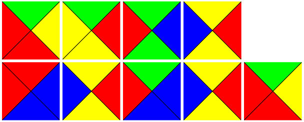

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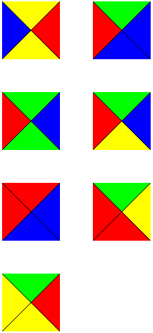

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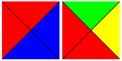

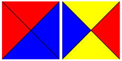

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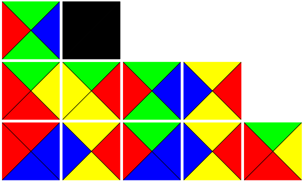
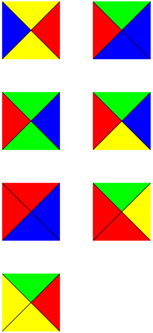
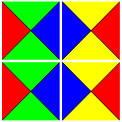
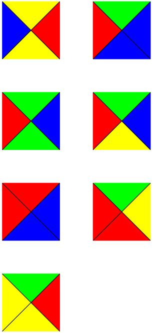


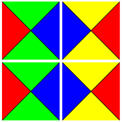
Wang tilings and transducers





Wang tilings and transducers

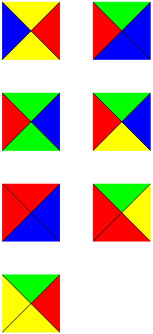


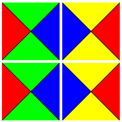


[Berger 1964]

The Domino Problem is undecidable.

Wang tilings and transducers



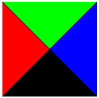


[Berger 1964]

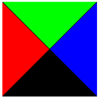
The Domino Problem is undecidable.

Key property: aperiodicity

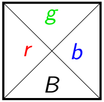
Wang tiling and transducers



Wang tiling and transducers



$\Leftrightarrow$



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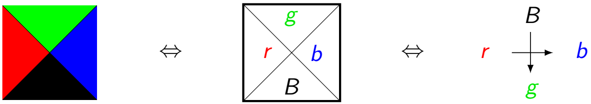

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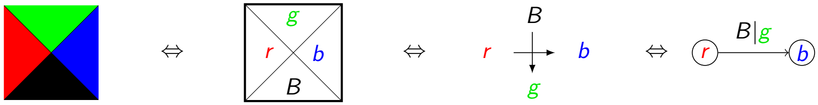

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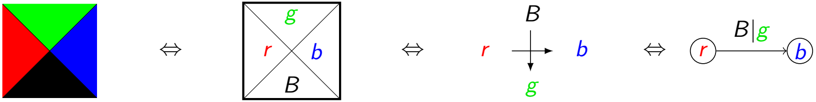
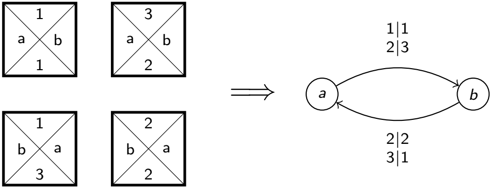
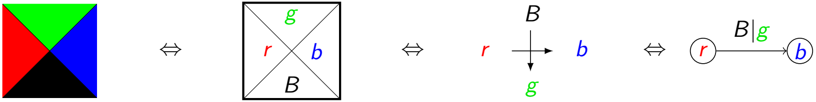
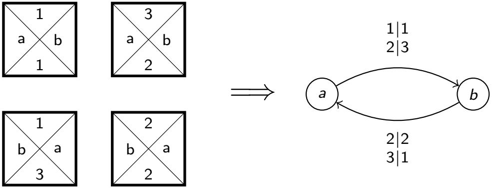




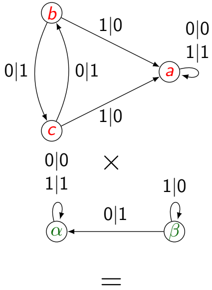
Wang tiling and transducers





- line of dominoes ⟺ run in the transducer
- multiple lines ⟺ transducer composition

Composition/Product of Transducers



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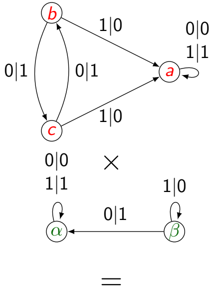
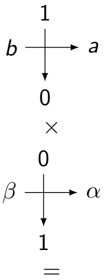
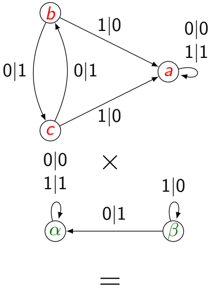
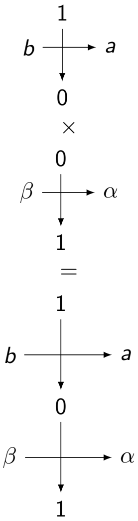


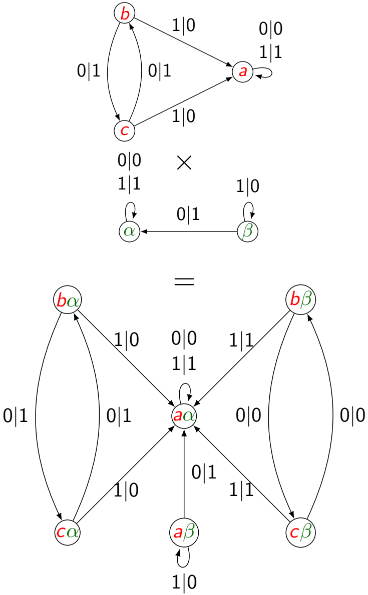


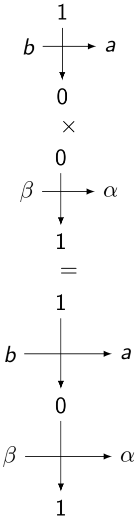
Composition/Product of Transducers



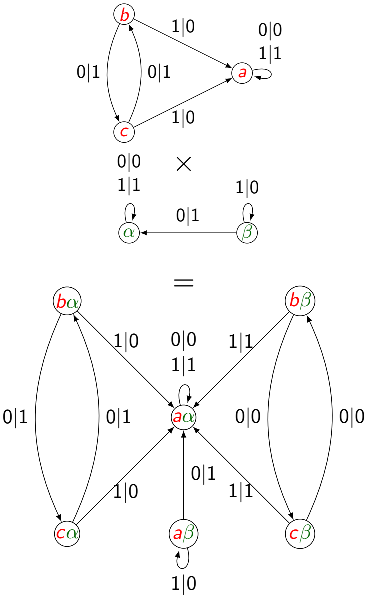


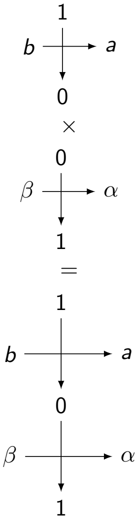
Composition/Product of Transducers

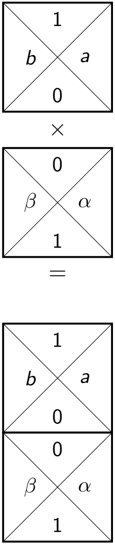




Composition/Product of Transducers



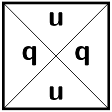




Find singular points

Lemma

$\exists\, \xi$ singular $\iff \exists \mathbf{u}^\omega$ singular



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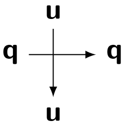
$\exists \mathbf{u}, \mathbf{q},\ \delta_{\mathbf{u}}(\mathbf{q}) = \mathbf{q},\ \rho_{\mathbf{q}}(\mathbf{u}) = \mathbf{u}$



Definition

$(\mathbf{q}, \mathbf{u})$ commuting pair

Find singular points

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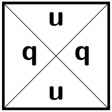

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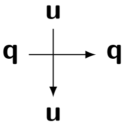

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Commuting pair and helix graph

How to find commuting pairs?

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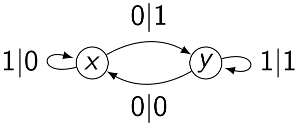

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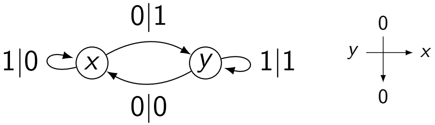

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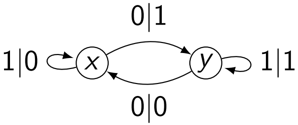

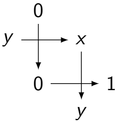

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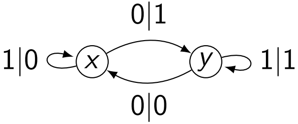

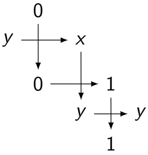

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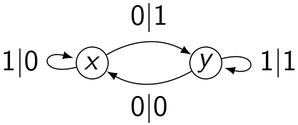

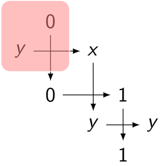

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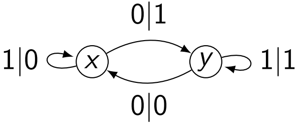

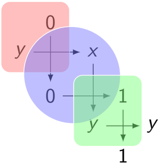

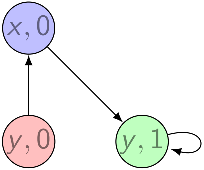

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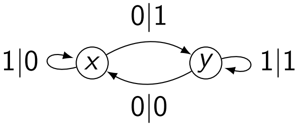

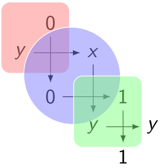

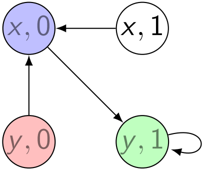

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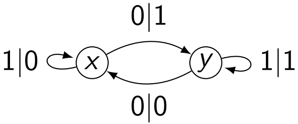

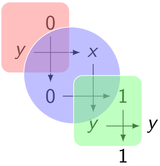

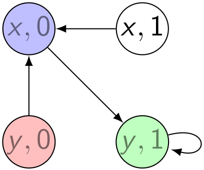

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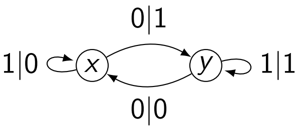

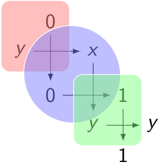

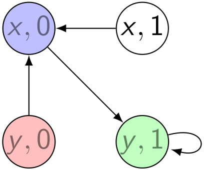

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Lemma

Tileset associated with Mealy automaton $\Longrightarrow$ periodic tiling

Restricted tilesets and undecidability

$(\mathsf{e}, \mathbf{u})$ is always a commuting pair but $\mathbf{u}^\omega$ is not always singular

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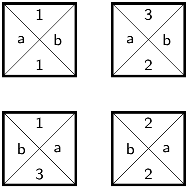

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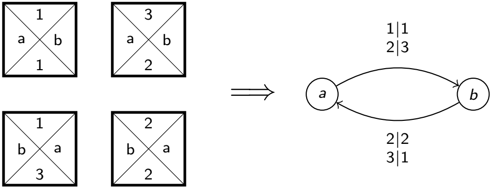

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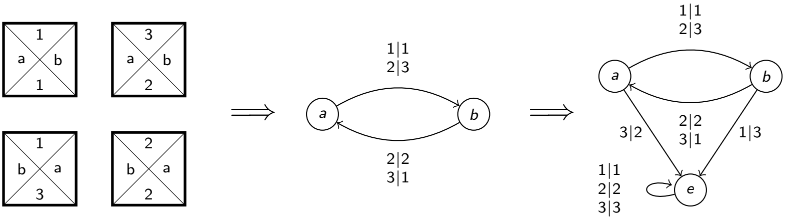


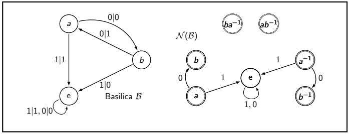
Characterising singular points

Theorem

The set of singular points has measure 0.

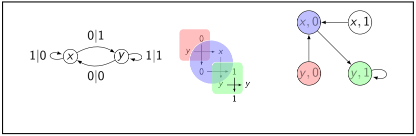
Expressing Sing as a language:

(fractal) contracting automata



Consider specific stabilisers, via commuting pairs:

(bi)reversible automata



Thanks

If you enjoyed that talk

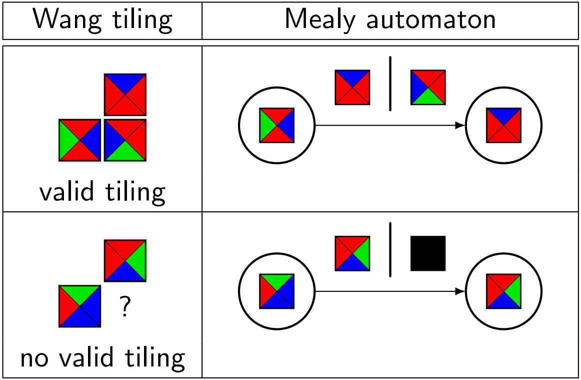
MealyM final event July 10–13

- Nathalie Aubrun
- Laurent Bartholdi
- Tara Brough
- Alan J. Cain
- Daniele D'Angeli
- Martin Delacourt
- Francesca Fiorenzi
- Pierre Gillibert
- Markus Lohrey
- Jean Mairesse
- Irène Marcovici
- Cyril Nicaud
- Nicolas Ollinger
- Emanuele Rodaro
- Dmytro Savchuk
- Reem Yassawi
- Matthieu Picantin's HdR defense
- my PhD defense

https://mealym.sciencesconf.org/

Thanks

Gillibert's connection between Wang tiling and Mealy automata



The Mealy automaton $\mathcal{W}_{\mathcal{T}}$ associated with a *wn*-deterministic Wang tileset $\mathcal{T}$ according to Gillibert.

Lemma

If $\mathbb{Z}^2$ admits a valid Wang tiling for $\mathcal{T}$, then $\langle \mathcal{A}(\mathcal{T}) \rangle_+$ is infinite.

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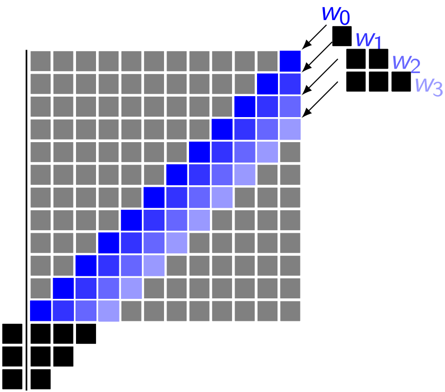

The action of the tile ■ corresponds to an element of infinite order.

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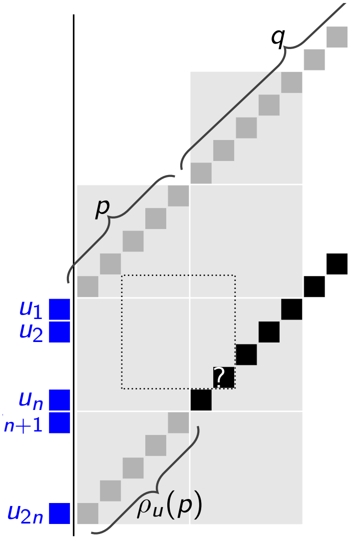

If some tile was not ■, the $n \times n$ square) could be tiled.

Theorem [Gillibert '13]

The semigroup $\langle\mathcal{A}(\mathcal{T})\rangle_+$ is infinite if and only if $\mathcal{T}$ can tile the discrete plane $\mathbb{Z}^2$.

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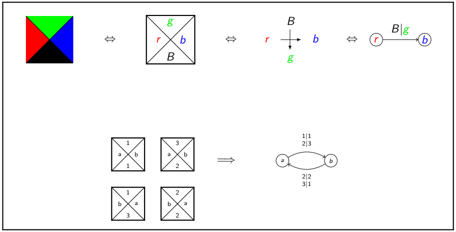
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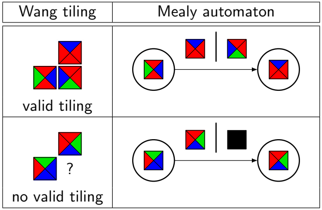
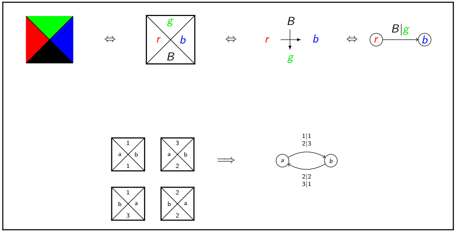
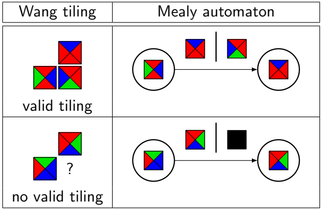
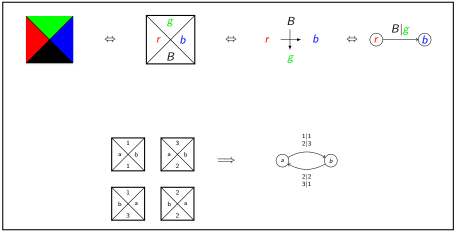
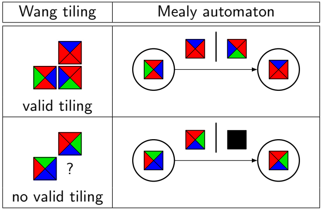

Wang tiling
Mealy automaton
valid tiling
?
no valid tiling





- Explore in depth these connections
- Find eventual new translations
- Specialize to certain subclasses





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