A CLASS OF NON-RIGID INTERVAL EXCHANGES

S. Ferenczi, P. Hubert

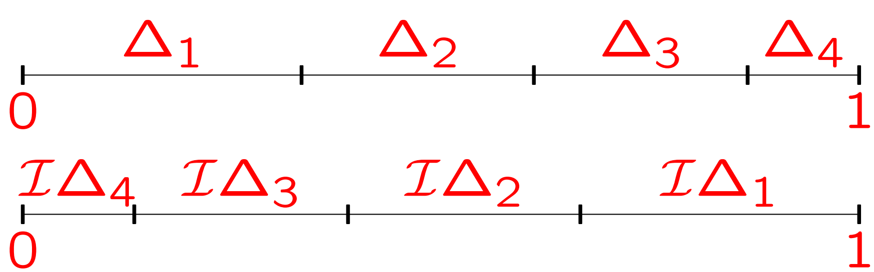
INTERVAL EXCHANGES

A k-interval exchange or k-iet $\mathcal{I}$ with probability vector $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, and permutation π is defined by

$$\mathcal{I}x = x + \sum_{\pi^{-1}(j)<\pi^{-1}(i)} \alpha_j - \sum_{j<i} \alpha_j.$$

when x is in the interval

$$\Delta_i = \left[\sum_{j<i} \alpha_j, \sum_{j \leqslant i} \alpha_j \right[.$$



CLASSICAL RESULTS

Almost all iet = for fixed π, for Lebesgue-almost all probability vector in the simplex.

If π is primitive, almost all iet are minimal = all orbits are dense (Keane, 1975, with explicit condition called i.d.o.c.).

If π is primitive, allmost all iet are uniquely ergodic = only one invariant probability, Lebesgue (Veech/Masur, 1982).

If π is not circular, almost all iet are weakly mixing = no eigenvalues with measurable eigenvectors (Avila - Forni, 2007).

LESS CLASSICAL RESULTS

Are almost all iet simple ?
Question of Veech (1982), negative answer for $k = 3$ by Chaika - Eskin (2017).

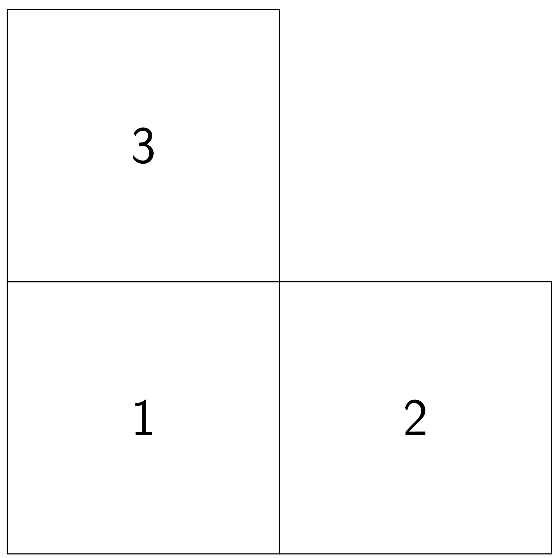
Almost all iet are rigid $=$ there exists a sequence $q_n \to \infty$ such that for any measurable set

$\mu(T^{q_n} A \triangle A) \to 0.$

Examples of non-rigid iet were known only for 3 intervals. Until Robertson (2017) and F-H.

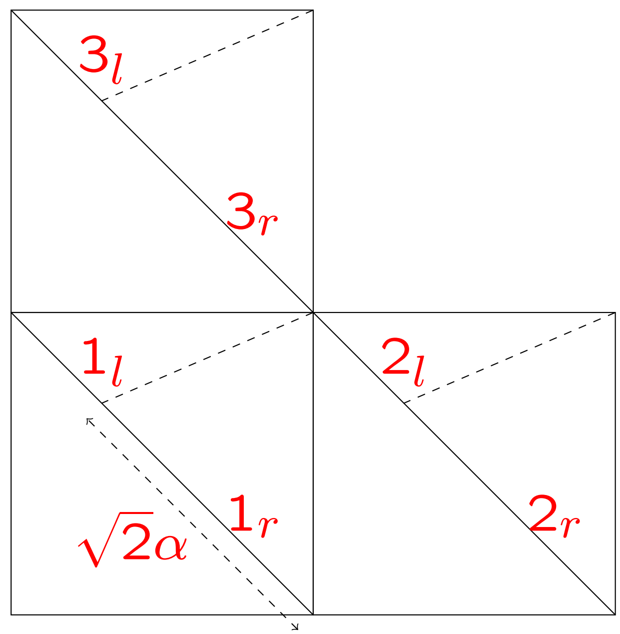
SQUARE TILED SURFACES

It is generated by d squares, whose sides are glued according to two permutations σ along the vertical and τ along the horizontal.
In this example $\tau(1, 2, 3) = (2, 1, 3)$ and $\sigma(1, 2, 3) = (3, 2, 1)$.



BUILDING AN IET

We take the directional flow of angle θ on a square tiled surface and its first return map on the union of negative diagonals. Let $\alpha = \frac{1}{1+\tan\theta}$.

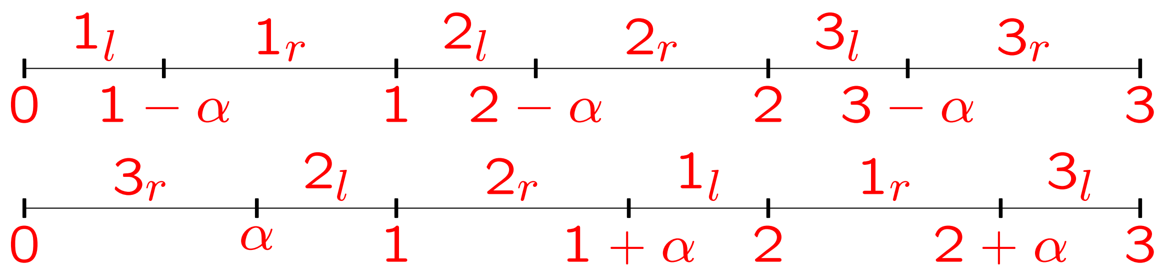


SQUARE TILED INTERVAL EXCHANGES

let where the departure intervals are

$[i, i+1-\alpha[$, denoted by i_l, sent to $[j+\alpha, j+1[$, $j = \tau i$,

$[i+1-\alpha, i+1[$, denoted by i_r, sent to $[j, j+\alpha[$, $j = \sigma i$.



RIGIDITY

We take α irrational.

If the square-tiled surface is connected, equivalently no strict subset of $\{1 \ldots d\}$ is invariant by σ and τ, the iet is minimal and uniquely ergodic.

Theorem 1. *If the square-tiled surface is of genus at least 2 (equivalently $\sigma\tau \neq \tau\sigma$), the associated iet is rigid if and only if α has unbounded partial quotients ; the linear flow in direction θ on X is rigid if and only if the slope $\tan\theta$ has unbounded partial quotients.*

The nontrivial part is the proof of non-rigidity if BPQ.

SYMBOLIC SYSTEMS

For two words of equal length $w = w_1 \dots w_q$ and $w' = w'_1 \dots w'_q$, Hamming distance $=$ $\bar{d}(w, w') = \frac{1}{q}\#\{i; w_i \neq w'_i\}$.

Symbolic system $=$ the shift on infinite sequences on a finite alphabet.

For two words of equal length $w = w_1 \dots w_q$ and $w' = w'_1 \dots w'_q$, Hamming distance $=$ $\bar{d}(w, w') = \frac{1}{q}\#\{i; w_i \neq w'_i\}$.

For a uniquely ergodic symbolic system, rigidity implies that for any infinite sequence $x_0x_1x_2...$ in the system, for a given ϵ, $n > n_0$, $N > n_1$,
$\bar{d}(x_0...x_N, x_{q_n}...x_{q_n+N}) < \epsilon.$

$\bar{d}$ - SEPARATION

$\bar{d}$-separated $=$ there exists C such that for any two words w and w' of length q produced by the system, if $\bar{d}(w,w') < C$, then $\{1,\dots q\} = I \cup J \cup K$, intervals in increasing order, $w_J = w'_J$, $\bar{d}(w_I, w'_I) = \bar{d}(w_K, w'_K) = 1$ except for empty I or K.

$\bar{d}$-separation was introduced by del Junco (1977) for the Thue - Morse sequence.

$\bar{d}$-separation and aperiodicity imply non-rigidity (Lemanczyk - Mentzen, 1988).

NON - RIGIDITY OF SQUARE TILED IETS

Our iet is $\bar{d}$-separated if and only if $\sigma\tau i \neq \tau\sigma i$ for all i.

A sequence in the symbolic system gives a Sturmian sequence, $u \to \phi(u)$ by $i_l \to l$, $j_r \to r$ for all i, j.

If we have $\sigma\tau \neq \tau\sigma$, then there exists C such that for d pairs of words of length q, if

— $\sum_{i=1}^{d} \bar{d}(v_i, v'_i) < C$,

— $\phi(v_i)$ is the same word u for all i,

— $\phi(v'_i)$ is the same word u' for all i,

— $v_i \neq v_j$ for $i \neq j$.

Then $\{1, \dots q\} = I \cup J \cup K$, intervals in increasing order

— $v_{i,J} = v'_{i,J}$ for all i,

— $\sum_{i=1}^{d} \bar{d}(v_{i,I}, v'_{i,I}) \geqslant 1$ if I is nonempty,

— $\sum_{i=1}^{d} \bar{d}(v_{i,K}, v'_{i,K}) \geqslant 1$ if K is nonempty,

This property contradicts rigidity.